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AN IMPROVEMENT ON THE UPPER BOUND OF THE NILPOTENCY CLASS OF SEMIDIRECT PRODUCTS OF p-GROUPS

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ABSTRACT. The semidirect product of a group A by a group B is necessarily nilpotent only in the case A and B are p-groups for the same prime p, A is nilpotent of bounded exponent, and B is finite. In an earlier paper Morley has established an upper bound on the class of a nilpotent semidirect product of an abelian p-group of bounded exponent by an arbitrary finite p-group. In this paper this result is improved by considering a direct product decomposition for B and also by extending the results to give a new upper bound on the class in the most general case. The standard wreath product of A by B is a nilpotent semidirect product of relatively large class in the case A and B satisfy the conditions above, and this new bound improves the known results on the class of these wreath products.

1. **Introduction.** A group which is a semidirect product of A by B can be assumed nilpotent only with the conditions that A and B are p-groups for the same prime number p, A is nilpotent of bounded exponent, and B is finite (Baumslag [1]). The standard wreath product of A by B contains all semidirect products of A by B and the exact class of A wr B has been given by Liebeck [3] in the case A and B are abelian p-groups, A is of bounded exponent and B is finite. Meldrum [4] has given the class of A wr B in the case A is nilpotent of exponent p and B is finite abelian. Morley [5] derives an upper bound for the class of a group which is a semidirect product of an abelian p-group of exponent p^{n+1} by an arbitrary finite p-group. In this paper, an improvement in the upper bound of [5] is established and the new bound is also extended to the most general case of a nilpotent semidirect product of p-groups. The improvement in the bound is accomplished by considering a direct product factorization of the group B.

2. **Notation and preliminary results.** The notation and definitions used in this paper agree with, but in cases generalize, those in [5]. $(g_1, \ldots, g_n)$ and $(g_1, (n-1)g_2)$ indicate commutator elements of length n, the second one having the last $n-1$ entries all g_2. An arbitrary ascending central series (see [2]) of the group G is denoted $G_0 < G_1 < \cdots$ and the well-known lower

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central series is written $G_{(i)}$, $i = 0, 1, 2, \ldots$. The nilpotency class of G is denoted $\mathrm{Cl}(G)$ and $\mathrm{Cl}(G) = L$ iff $G_{(L+1)} = E$ and $G_{(L)} \neq E$, E the trivial subgroup containing the identity element only. If H_i, $i = 1, \ldots, n$, are subgroups of G, then $(H_1, \ldots, H_n)$ denotes the subgroup of G generated by $\{(h_1, \ldots, h_n) | h_i \in H_i\}$. The well-known commutator identities

$$(2.1) \qquad (x,yz) = (x,z)(x,y)(x,y,z), \qquad (xy,z) = (x,z)(x,z,y)(y,z),$$

are used in the proof of Theorem 3.1.

For an extension W of A by B which is a semidirect product we assume both A and B are subgroups of W.

If $B = B_1 \times B_2 \times \cdots \times B_m$ is a direct product of finite p-groups, then $B_{k,0} < \cdots < B_{k,L(k)}$ denotes an ascending central series of B_k contained in B. Since $B_{k,i}/B_{k,i-1}$ is a finite abelian p-group, $B_{k,i}$ contains a minimal independent set of generators modulo $B_{k,i-1}$. The elements of such a generating set are written $\{b_{k,i,j}\}$, $j = 1, \ldots, r(k,i)$, $r(k,i)$ being the cardinality of the generating set which is called the rank of the factor group.

DEFINITION 2.2. Let $p(k,i,j)$, $1 \leqslant j \leqslant r(k,i)$, denote the descending prime power orders of the cyclic groups in the decomposition of $B_{k,i}/B_{k,i-1}$, $k = 1, \ldots, m$ and $i = 1, \ldots, L(k)$, for a specified ascending central series of length $L(k)$ for B_k. Define $L = \max_{1 \leqslant k \leqslant m} L(k)$ and for $L(k) < i \leqslant L$ set $p(k,i,j) = 1$ and $r(k,i) = 0$. Then we define

$$\lambda_{ki} = \sum_{j=1}^{r(k,i)} (p(k,i,j) - 1), \qquad 1 \leqslant k \leqslant m \text{ and } 1 \leqslant i \leqslant L,$$

$$d(k,t,s) = \prod_{j=t+1}^{s} p(k,j,1), \qquad 1 \leqslant k \leqslant m,\ 1 \leqslant s \leqslant L, \text{ and } 0 \leqslant t \leqslant s,$$

and

$$P_k(y_1, \ldots, y_s) = \sum_{t=1}^{s} d(k,t,s) y_t \quad \text{for } s = 1, 2, \ldots, L.$$

The multivariable linear polynomials $P_k(y_1, \ldots, y_s)$ have coefficients determined by the exponents of the factor groups $B_{k,i}/B_{k,i-1}$ and the λ_{ki} are dependent upon the complete cycle structure of these factor groups.

3. **The upper bound.** In the theorems of this section $B = B_1 \times \cdots \times B_m$ is a direct product of finite p-groups. The terms expressed are defined using arbitrary but specified ascending central series for the respective direct factors of B using Definition 2.2.

THEOREM 3.1. *Let W be a semidirect product of A by B, A abelian of exponent p^{n+1}. Then*

$$\mathrm{Cl}(W) \leqslant \sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + n(p-1)p^{-1} \max_{1 \leqslant k \leqslant m} d(k,0,L(k)) + 1.$$

PROOF. Let

$$c = \sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + n(p-1)p^{-1} \max_{1 \leqslant k \leqslant m} d(k, 0, L(k)) + 1.$$

We will use induction on m. The result for $m = 1$ is given in [5, Theorem 4.10]. So assume that the result holds for $m - 1$.

Assume $\mathrm{Cl}(W) > c$ and obtain a contradiction. Without loss of generality we assume $\mathrm{Cl}(W) = c + 1$ and choose $e \neq w \in W_{(c+1)}$. By [5, Corollary 3.4], $c > L = \mathrm{Cl}(B)$ and $w = (x, b_0, b_1, \ldots, b_{c-q})$ where $b_0 \in B_{(q)}$ and $b_i \in B$, $1 \leqslant i \leqslant c - q$ and $1 \leqslant q \leqslant L$. By the basic commutator identities, using the fact that A is abelian and $W_{(c+2)} = E$, the map $b_i \to (x, b_0, b_1, \ldots, b_{c-q})$ is a homomorphism for each i, $0 \leqslant i \leqslant c - q$. So we may assume that $b_i \in B_k$ for some $k = k(i)$, $0 \leqslant i \leqslant c - q$. Since $(B_k, B_l) = E$ for $k \neq l$, [5, Lemma 4.6] allows us to assume that all the elements b_i from a given B_k follow each other. Let the number of entries from B_k be c_k. Then $c = \sum_{k=1}^{m} c_k$.

Without loss of generality we may assume that

$$c_m > P_m(\lambda_{m1}, \ldots, \lambda_{mL(m)}) + (n - t_m)(p-1)p^{-1} d(m, 0, L(m)),$$

where t_m is minimal subject to this inequality holding. By [5, Theorem 4.10] applied to $A \cdot B_m$, we may assume that $(x, b_1, \ldots, b_{c_m})$ has order dividing p^{t_m}, where $b_i \in B_m$, $1 \leqslant i \leqslant m$.

If $t_m = 0$, then $w = e$ and the contradiction is obtained. So assume that $t_m \neq 0$. Then, by the minimality of t_m,

$$c' = c - c_m \geqslant \sum_{k=1}^{m-1} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + (t_m - 1)(p-1)p^{-1} \max_{1 \leqslant k \leqslant m-1} d(k, 0, L(k)) + 1.$$

Let $w' = (x', b_{c_m+1}, \ldots, b_c)$ where $x' = (x, b_1, \ldots, b_{c_m})$. By the induction hypothesis on m, $w' = e$ since $w' \in A \cdot (B_1 \times \cdots \times B_{m-1})$. This gives the final contradiction.

The proof of the following theorem is an adaption of the proof of Theorem 5.12 of [4].

THEOREM 3.2. *Let W be a semidirect product of A by $B = B_1 \times \cdots \times B_m$, A a nilpotent p-group of class c and B_k a finite p-group for each $k = 1, \ldots, m$. Suppose $A_0 < A_1 < \cdots < A_c$ is an ascending central series of A. If A_j/A_{j-1} has exponent $p^{n(j)}$ for $1 \leqslant j \leqslant c$, then*

$$\mathrm{Cl}(W) \leqslant c\left(\sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)})\right) + \left(\sum_{j=1}^{c} (n(j) - 1)\right)(p-1)p^{-1} \max_{1 \leqslant k \leqslant m} d(k, 0, L(k)) + c.$$

PROOF. We proceed by induction on c. Theorem 3.1 is just the statement of this result for $c = 1$ so we let $c > 1$. Define

$$t(j) = \sum_{k=1}^{m} P_k(\lambda_{k1}, \ldots, \lambda_{kL(k)}) + (n(j) - 1)(p - 1)p^{-1} \max_{1 \leqslant k \leqslant m} d(k, 0, L(k)) + 1,$$

Now A_1 is a normal subgroup of W and W/A_1 is a semidirect product of A/A_1 by B. By the induction hypothesis $\mathrm{Cl}\,(W/A_1) \leqslant \sum_{j=2}^{c} t(j)$ since $(A_j/A_1)/(A_{j-1}/A_1)$ is isomorphic to A_j/A_{j-1} for $2 \leqslant j \leqslant c$. Thus we have that $W_{(t)} \subseteq A_1$ for $t = \sum_{j=2}^{c} t(j) + 1$. The result now follows from Theorem 3.1 since A_1 is contained in the centre of A implies $(A_1, kW) \subseteq (A_1, kB)$.

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