ON THE FRACTIONAL PARTS OF $n/j, j = o(n)$

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Abstract. Dirichlet's result that if $J(n) = o(n)$ but $n^{1/2} = o(J(n))$, the numbers $n/j$ for $j = 1, \ldots, J(n)$ are nearly uniformly distributed modulo 1 (with error $\to 0$ as $n \to \infty$) is extended, $n^{1/2}$ being replaced by $n^\alpha$ for any $\alpha > 0$.

1. To illustrate the problem considered here (and the results): for large $n$, the real numbers $n/j$ for $j = 1, 2, \ldots, [n^{1/2}]$, reduced modulo 1, are nearly uniformly distributed. That is, for $\epsilon \in (0, 1)$, the fraction of those numbers $n/j$ that lie between $[n/j]$ and $[n/j] + \epsilon$ differs from $\epsilon$ by at most $\epsilon(n)$, where $\epsilon(n) \to 0$ as $n \to \infty$.

If $n^{1/2}$ is replaced by any function $J(n)$ satisfying $J(n) = o(n)$, but $n^{1/2} = o(J(n))$, the near-uniform distribution of those numbers is a result of Dirichlet (see [1, p. 327]), who showed also that the distribution is not uniform if $J(n) \neq o(n)$. This paper replaces Dirichlet's exponent $1/2$ by any $\alpha > 0$.

Much the hardest part of the proof is due to A. Walfisz, who proved a lemma on the distribution of some of the numbers in question. In 1932 Walfisz applied his lemma to estimates of the number of lattice points in an ellipsoid [2]; in 1963 he gave some other applications [3]. But this application, which seems the most natural one, also seems never to have been done.

Theorem. If $J(n) = o(n)$ but some $n^\alpha, \alpha > 0$, is $o(J(n))$, then the fraction of the first $[J(n)]$ numbers $n/j$ (mod 1) which lie in an interval of length $\epsilon$ in $R/Z$ differs from $\epsilon$ by at most $\epsilon$, where $\epsilon \to 0$ as $n \to \infty$.

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2. Walfisz's lemma concerns sums of the complex numbers $e(n/j) = \exp(2\pi in/j)$. The following specialization will suffice.

Lemma (Walfisz). Let $r$ be a positive integer, $w \in [0, 1]$, $R = 2^{-1}, R_1 = R + 1$, $M$ between $n^{1/(r+2)}$ and $n^{2/(r+3)}$ ($n$ need not be an integer). Then

$$\sum_{j=M}^{2M} e(n/(j + w)) = O(M^{-1/(r+1)} n^{1/R_1} \log n).$$

All we want of the concluding expression is that it is $o(M)$. Actually it is not, for the smallest $M$ allowed, viz. $n^{1/(r+2)}$; there the estimate is
$O(M \log M)$ (and is, of course, worthless). We narrow the requirements by adding

\[(*) \quad M > n^{1/[(r+2)].}\]

Still the different values of $r$ give overlapping intervals from $2n^{1/2}$ down through all $N^a$ in which the average value of $M + 1$ successive terms $e(n/(j + w))$ beginning at $j = M$ is always (as a function of $n$) $o(1)$. More precisely, for each $r$, if $n$ is large enough, each of those averages is less than $\varepsilon$ (by Walfisz’s proof and (*)).

$2n^{1/2}$ is not big enough (to adjoin Dirichlet’s case). However, the $o(1)$ conclusion extends all the way up $o(n)$.

**Corollary.** For $M(n) > n^{1/3}$, $M(n) = o(n)$, the average of $e(n/j)$ as $j$ goes from $M(n)$ to $2M(n)$ is $o(1)$.

**Proof.** First, if we replace some $n$ by $n' = n/M(n)$, $n'$ will still go to infinity with $n$ and “$o(1)$” may be referred equally well to varying $n$ or varying $n'$. This still applies though, precisely, we introduce $b(n) = 1 + [M(n)^2/n]$ and put $n^* = n/b(n)$, so that $n/j = n^*(j/b(n))$. Unless $M(n) > n^{1/2}$, $b(n) = 1$ and we did nothing. Otherwise (with negligible error) we replace $M$ by $M^* = (n^*)^{1/2}$. Precisely, add at most $b(n) - 1$ terms to the sequence to make its length a multiple of $b(n)$ (affecting the average by less than $b(n)/M(n) = o(1)$). The sequence of expressions $e(n^*/(j/b(n)))$ decomposes into $b(n)$ sequences in which denominators form progressions with difference $1$; each has, with error $o(1)$, the form in the lemma. So each has average $o(1)$, and the average of those $b(n)$ averages is still $o(1)$.

3. We wish to apply Weyl’s criterion, familiar in this form: a sequence $(a_j)$ of complex numbers of modulus 1 is uniformly distributed if, for $k = 1, 2, \ldots$, the average of the first $n$ $k$th powers $a_j^k$ approaches 0 as $n \to \infty$. We need the following form.

**Lemma (Weyl).** For each $\varepsilon > 0$ there exists $N$ such that given a finite family of complex numbers of modulus 1, if for $k < N$ the average of $a_j^k$ has modulus less than $1/N$, then the fraction of the $a_i$ which lie in an interval on the unit circle of length $2\pi\varepsilon$ is between $t - \varepsilon$ and $t + \varepsilon$.

Supposing this false, for some $\varepsilon$ we should have a sequence of examples, $N \to \infty$, missing by $\varepsilon$ on certain intervals. For a subsequence, the intervals converge to a limit, and it is simple routine to patch together an infinite sequence $(a_j)$ violating the criterion as previously stated, which is absurd.

The theorem follows. For, first, for $M > n^a$ but $o(n)$, the average of $e(n/j)$ for $j$ from $M$ to $2M$ is small (for large $n$). Also $M$ exceeds $(2n)^{a/2}$, and the average of $e(2n/j)$ for those $j$ is small. This is true of $e(kn/j)$, uniformly in $k$, as long as $k < n$, $(kn)^{a/2} < n^a$. So the modified Weyl criterion tells us that those $e(n/j)$ are uniformly distributed to within $\varepsilon$ ($\varepsilon \to 0$ as $n \to \infty$). To within $2\varepsilon$, we can apply this to all $j$ less than $J(n)$, for $J(n) > 2n^a$ but $J(n) = o(n)$. Let $M_0 = [2^{-s}J(n)]$, $M_i = 2M_{i-1}$ for $i < s$; the $s$ intervals $[M_i, 2M_i]$ reach from 1 to $J(n)$, with negligible error and negligible overlap, and all have $e(n/j)$ uniformly distributed (within $\varepsilon$).
REFERENCES


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