COHOMOLOGY OF ASSOCIATIVE TRIPLE SYSTEMS

RENATE CARLSSON

Abstract. In the following a cohomology for associative triple systems is defined. The classical corollaries can be extended to the associative triples.

1. Introduction. One distinguishes two associative triple systems. The associative triples of the first kind or ternary rings were introduced by Lister [6]. Those of the second kind or Hestenes ternary rings were first investigated by Hestenes and as special alternative triples by Loos [7]. Typical examples for the latter are the $p \times q$ matrices in an associative algebra with an involution, the triple product given by $(x, y, z) \mapsto x'y'z$, $'y$ the transposed-conjugate matrix.

In this note we define a canonic cohomology for associative triples. Cohomologies are well known for certain algebras and Lie triples [2]. The corollaries for associative triples are analogous to the classical results.

In the following $K$ is a commutative and associative ring with unit 1, the modules taken unitary over $K$. If $K$ is explicitly assumed as a field then the dimensions over $K$ are finite.

2. Definitions. The universal $(i)$-imbedding. A $K$-module $A$ with a trilinear inner composition $A \times A \times A \to A$, with $(x, y, z) \mapsto \langle xyz \rangle$, is an associative triple system of the 1st kind if

$$\langle (xyz)uv \rangle = \langle x(yzu) v \rangle = \langle xy(zuv) \rangle$$

for $x, y, z, u, v \in A$. $A$ is an associative triple of the 2nd kind, when

$$\langle (xyz)uv \rangle = \langle x(uzy) v \rangle = \langle xy(zuv) \rangle.$$  

A trimodule $M$ over a triple system $A$ is a $K$-module $M$ together with 3 trilinear compositions $A \times A \times M \to M$, the products denoted $\langle xym \rangle$, $\langle xmy \rangle$, $\langle mxy \rangle$, where $m \in M$. The split null extension or semidirect sum $A + M$ of $M$ over $A$ is the $K$-module direct sum with

$$\langle (x + m_i)(y + m_j)(z + m_3) \rangle = \langle xyz \rangle + \langle m_1yz \rangle + \langle xm_2z \rangle + \langle xym_3 \rangle$$

for $x, y, z \in A$, $m_i \in M$, $i = 1, 2, 3$. A trimodule $M$ is associative of the 1st kind over $A$ if the semidirect sum is an associative triple of the 1st kind, equally for the 2nd kind. In the following $A$ always denotes an associative triple, and $M$ an associative trimodule over $A$ of the same kind if nothing...
other is stated, where associative means associative of the 1st or 2nd kind. For K-modules N, P let \( N \oplus P \) denote the K-module direct sum.

Let \( B \) be an associative algebra, the multiplication designed by \( " \circ " \), and \( \iota \): \( A \to B \) a K-module monomorphism. \( (B, \iota) \) is an imbedding of an associative triple \( A \) of the 1st kind if \( B = \iota A + \iota A \circ \iota A \) with \( \iota x \circ \iota y \circ \iota z = \langle xyz \rangle \) \( x, y, z \in A \), and the imbedding is direct if the sum is direct \([6]\). \( (B, \iota) \) is universal if for each homomorphism \( f: A \to A' \) onto a triple \( A' \), and any imbedding \( (B', \kappa) \) of \( A' \) with monomorphism \( \kappa \) there exists a unique algebra homomorphism \( f': B \to B' \) onto \( B' \) so that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow & & \downarrow \\
B & \xrightarrow{f'} & B'
\end{array}
\]

is commutative, \( \iota \) denoting a K-module monomorphism.

Let \( A \) be associative of the 2nd kind, \( j \) an involution of \( B \) with \( \overline{A} = j(\iota A) \), \( \overline{b} = j(b), b \in B \). \( (B, \iota, j) \) is a j-imbedding of \( A \) if \( B = \iota A + \iota A + \iota A \circ \iota A + \iota A \circ \iota A \) with \( \iota A \circ \iota A = \{0\} \) and \( \iota x \circ \iota y \circ \iota z = \langle xyz \rangle \) for \( x, y, z \in A \). The j-imbedding is direct if the above sum is direct. \( (B, \iota, j) \) is universal if for any homomorphism \( f \) onto \( A' \), \( (B', \kappa, j') \) a j'-imbedding of \( A' \), there exists a unique homomorphism \( f': B \to B' \) so that the above diagram is commutative and \( j'f = f \). The standard imbedding of \( A \) is a direct imbedding if \( A \) is associative of the 1st kind \([6]\), and it contains a direct j'-imbedding otherwise, as is easily noticed (cf. \([7]\)). One immediately has

**Theorem 1.** Let \( A \) be an associative triple over \( K \). Then there exists a universal imbedding \( (U(A), \iota) \), respectively a universal j-imbedding \( (U(A), \iota, j) \) of \( A \). If \( (V(A), \kappa) \), respectively \( (V(A), \kappa, j') \), is a universal (j'-) imbedding too, then there is a unique isomorphism \( g \) of \( U(A) \) onto \( V(A) \) with \( \kappa = gi \).

**Proof.** Let \( T(A) := K \cdot 1 \oplus T^*(A) \) be the tensor algebra of the \( K \)-module \( A \) with \( T^*(A) := \bigoplus (\otimes^n A), n \in \mathbb{N} \). Then \( T(A) \) is a universal enveloping associative algebra for \( A \) with unit 1. We construct \( U(A) \) as a factor algebra: In case of the 2nd kind let \( \overline{A} \) be a \( K \)-module isomorphic to \( A \), the isomorphism given by \( x \mapsto \overline{x} \). Let \( Q \) denote the ideal of \( T^*(A \oplus \overline{A}) \) generated by \( x \otimes y, \overline{x} \otimes \overline{y}, x \otimes \overline{y} \otimes z - \langle xyz \rangle, \overline{x} \otimes y \otimes \overline{z} - \langle zyx \rangle \) for \( x, y, z \in A \). Set \( U(A) := T^*(A \oplus \overline{A})/Q \) and \( \iota: x \mapsto x + Q \). Observing the elementary properties of the tensor product one notes the existence of an involution \( j \) of \( U(A) \) with \( j(x + Q) = \overline{x} + Q \). Thus \( (U(A), \iota, j) \) is a universal j-imbedding of \( A \). For the 1st kind take \( T^*(A) \) modulo the ideal \( Q \) generated by the elements \( x \otimes y \otimes z - \langle xyz \rangle \).

The proof of the second part of the statement is standard. \( \square \)

In the subsequent we may suppose that \( \iota \) is the inclusion map. For an algebra \( C \) let \( C^{[n]} \) denote the \( n \)-fold cartesian product. Let \( U(A) \) design a universal (j-) imbedding of \( A \). Obviously \( U(A) \) is direct.

For a direct imbedding \( B = A \oplus A \circ A \) let \( \varepsilon_1: B \to A, \varepsilon_2: B \to A \circ A \) be
the canonic projections. If $B$ is a direct imbedding let $\varepsilon_1, \varepsilon_2, \varepsilon_{-1}, \varepsilon_{-2}$ be its canonic projections onto $A, A \circ A, A, A \circ A$. Set $B_k := \varepsilon_k(B), k \in J := \{1, 2\}$, respectively $J := \{-2, -1, 1, 2\}$. The formal multiplication of the $B$-components $B_k$ is obviously declared, e.g. for the 2nd kind $A \circ (A \circ A) := A, (A \circ A) \circ (A \circ A) := A \circ A, A \circ A := \{0\}, \ldots$. Similar to the case of Lie triples [2] we will consider some homogeneous multilinear mappings of universal ($\cdot$-) imbeddings. For $x := (x_1, \ldots, x_n) \in B^{[n]}$ with $x \in B$, $k = k(i)$, the $B$-component is $\Phi(x) := \prod B_k(i), i = 1, \ldots, n$, the product formal. Let $C$ be a subalgebra of $B$ with $\varepsilon_k(C) \subset C$, and $B'$ a ($j'$-) imbedding of $A'$ with $A'$ of the same kind as $A$. Then $f: C^{[n]} \to B'$ is isovarying if

$$\Phi(x) \subset B_k \Rightarrow f(x) \subset B'_k,$$

and antivarying if for $\Phi(x) \neq \{0\}$

$$\Phi(x) \subset B_k \Rightarrow f(x) \subset \bigoplus B'_l, \quad l \neq k, l \in J.$$

Any $f$ has a unique decomposition $f = f_1 + f_2$, $f_1$ isovarying and $f_2$ antivarying. Denote $\varepsilon: f \mapsto f_1$. Then $\varepsilon^2 := \varepsilon \cdot \varepsilon = \varepsilon$.

Let $B := A + M$, and let $J$ be the ideal in $U(B)$ equal to

$$\{ a \in B \circ M + M \circ B | a \circ a = A \circ a = \{0\} \},$$

respectively in the case of the 2nd kind

$$\{ a \in B \circ \bar{M} + \bar{M} \circ B \circ \bar{M} + \bar{M} \circ B | a \circ (A + \bar{A}) = (A + \bar{A}) \circ a = \{0\} \}.$$

Then the universal module imbedding $M_U$ is the ideal of $U(A + M)$ generated by $M$, respectively $M \oplus \bar{M}$, in the case of 2nd kind associativity, modulo $J$. The maps corresponding to $\varepsilon$ and $\varepsilon, j$ for the 2nd kind after the passages from $U(A + M)$ to $U(A)$ and $M_U$ are denoted equally. $M_U$ is an associative bimodule over $U(A)$ in the obvious way.

3. The cohomology. Let $C^n$ be the linear space of the $n$-cochains of $U(A)$ for $M_U$ [3], $n \in \mathbb{N}_0$. These are the $n$-linear mappings of $U(A)^{[n]}$ into $M_U$ for $n \in \mathbb{N}$, while $C^0 = M_U$. The coboundary operator is defined by

$$\delta f(a_1) = a_1 \cdot f - f \cdot a_1, \quad \text{if } f \in C^0,$$

$$\delta f(a_1, \ldots, a_{n+1}) = a_1 \cdot f(a_2, \ldots, a_{n+1})$$

$$+ \sum_{i=1}^n (-1)^i f(a_1, \ldots, a_i \circ a_{i+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1}$$

for $f \in C^n, n \neq 0, a_i \in U(A)$.

$Z^n, B^n$ denote the $K$-modules of the $n$-cocycles and the $n$-coboundaries where $B^0 := \{0\}$. $H^n(U(A), M_U) := Z^n/B^n$ is the $n$-dimensional cohomology group. If $x = (x_1, \ldots, x_n) \in U(A)^{[n]}$, $n \neq 0$, $U(A)$ a $j$-imbedding, set $j(x) := (j(x_1), \ldots, j(x_n))$. For $f \in C^0$ extend $\varepsilon$ by $\varepsilon(f) := \varepsilon_1 f$ if $A$ is of the 1st kind, and if $A$ is of the 2nd kind let $\varepsilon(f) := (\varepsilon_2 + \varepsilon_{-2}) f, fj := f$. Then the $K$-module
of the n-cochains of the associative triple system A for M is
\[ C^n(A, M) := \{ f \in C^n | \epsilon(f) = f \} \]
for \( n \in \mathbb{N}_0 \) if A and M are associative of the 1st kind. In case of the 2nd kind define
\[ C^n(A, M) := \{ f \in C^n | \epsilon(f) = f, j^f = (-1)^{n(n+1)/2 + 1} f \} \]
for \( n \in \mathbb{N}_0 \) if \( \frac{1}{2} \in K \). If \( 2 \cdot K = \{0\} \) modify this definition by \( C^0(A, M) := \{ f \in C^0 | \epsilon(f) = f, j^f = f + z \text{ with } z \in M, a \cdot z = z \cdot a \text{ for all } a \in U(A) \} \).

The coboundary operator \( \partial \) is the restriction of \( \delta \) from \( C^n \) onto \( C^n(A, M) \). The modules \( C^n(A, M) \) together with \( \partial \) define a cochain complex. \( Z^n(A, M) \) is the submodule of n-cocycles and \( B^n(A, M) \) of n-coboundaries. Let \( H^n(A, M) \) be the n-dimensional cohomology group with \( H^n(A, M) := Z^n(A, M) / B^n(A, M) \). Set \( Z^n_+ := Z^n \cap C^n(A, M) \) and let \( B^n_+ := B^n \cap C^n(A, M) \). We have

**Lemma 1.** Let A be an associative triple system, M an associative trimodule over A, and \( \frac{1}{2} \in K \) in case of the 2nd kind. Then
\[ H^n(A, M) = Z^n_+/B^n_+, \text{ and } H^n(U(A), M_U) \cong H^n(A, M) \oplus H(n), \]
\( H(n) \) a submodule of \( H^n(U(A), M_U), n \in \mathbb{N}_0 \).

**Proof.** Let A and M be associative of the 2nd kind, and \( \frac{1}{2} \in K \). Then for \( f \in C^n \), we have \( f = \epsilon(f) + (f - \epsilon(f)) \) and \( f = \frac{1}{2} \cdot ((f + j^f) + (f - j^f)) \). These are the decompositions of \( f \) into eigenfunctions for the endomorphisms \( \epsilon \) and \( j^f \). An easy verification shows that \( \epsilon \) and \( j^f \) commute. Thus there is a weight space decomposition \( C^n = \oplus C^a_{n, \beta} \), \( \alpha \) the root of \( \epsilon \), \( \beta \) the root of \( j^f \) for \( C^a_{n, \beta} \). Now \( C^n(A, M) = \oplus C^a_{n, \beta} \). Since \( \delta : C^a_{n, \beta} \to C^a_{n-1, \beta} \cdot (-1) \cdot (n+1)/2 \cdot \beta \), we get \( B^n(A, M) = B^n_+ \). The rest is obvious. For the case of the 1st kind one proceeds similarly and even more immediately.

The associative triple A is separable over the field K, if for every base field extension \( K \) of \( K \) the base field extension \( A \) of \( A \) is semisimple. Note that as in the binary case the radical of a ternary algebra \( A \) coincides with the radical of the ternary ring \( A \) when \( \frac{1}{2} \in K \) [8]. We receive

**Corollary 1.** Let M be an associative trimodule of the 1st or 2nd kind over A, and A separable over a field K, \( \text{char}(K) \neq 2 \). Then for all \( n \in \mathbb{N}, H^n(A, M) = \{0\} \).

**Proof.** Because A is semisimple the standard imbedding of A is semisimple too by [6], respectively [7]. Any semisimple imbedding, respectively j-imbedding, is isomorphic to \( U(A) \). The isomorphism follows as the corresponding one in case of Lie triples [4]. Thus \( U(A) \) is separable and
\[ H^n(U(A), M_U) = \{0\} \text{ for all } n \in \mathbb{N}. \]

Corollary 1 includes the statements corresponding to the two Whitehead lemmas.

A linear mapping \( D : A \to M \) is a derivation of A in M if
\[ D(\langle xyz \rangle) = \langle D(x)yz \rangle + \langle xD(y)z \rangle + \langle xyD(z) \rangle, \quad x, y, z \in A. \]
Inner derivations are those generated by the triple multiplications in the semidirect sum \( A + M \).

If \( \text{Der}(A, M) \) is the \( K \)-module of derivations of \( A \) in \( M \), then we have \( Z^1(A, M) \cong \text{Der}(A, M) \), the elements of \( B^1(A, M) \) related to inner derivations of \( A \) in \( M \). As isomorphism of \( Z^1(A, M) \) onto \( \text{Der}(A, M) \) we may choose the restriction of the elements \( f \in Z^1(A, M) \) from \( U(A) \) onto \( A \). It follows immediately that in case of the 2nd kind, \( \text{char}(K) = 2 \) and \( n = 1 \), Corollary 1 is valid too. Hence

**Corollary 2.** Let \( A \) and \( M \) be associative of the same kind, \( A \) separable over the field \( K \), and \( \text{char}(K) \neq 2 \) in case of the 1st kind. Then every derivation of \( A \) in \( M \) is inner.

We may remark that in case of \( \text{char}(K) = 2 \) the trivial 1st kind triple \( A := \{0, 1\} \) with \( \langle 111 \rangle = 1 \) is separable, while the radical of \( U(A) \) is \( \{0, 1 + 1 \circ 1\} \neq \{0\} \).

Now let \( C \) denote a binary algebra, \( N \) a \( C \)-bimodule. A semiautomorphism \( f \) of \( N \) is an algebra automorphism of the semidirect sum \( C + N \) with \( f(A) \subset A, f(N) \subset N \). It holds:

**Lemma 2.** Let \( G \) be a finite group of semiautomorphisms of a bimodule \( N \) over an algebra \( C \), \( n \) the order of \( G \), and \( 1/n \in K \). If \( N = Q \oplus P \), \( Q \) and \( P \) are \( C \)-submodules with \( G(Q) = Q \), then there exists a \( C \)-submodule \( R \) of \( N \) with \( G(R) = R \) and \( N = Q \oplus R \).

The proof follows considering the semidirect sum \( C + N \) and going on along the lines of [10].

Irreducible trimodules, complete reducibility, triple (trimodule) homomorphisms are defined as obvious. If \( A \) is semisimple, \( Z \) the center of \( U(A) \), and \( Z \otimes Z \) semisimple, \( A \) is \( c \)-semisimple. Denote \( W := U(A) \otimes (U(A)^op) \), \( U(A)^op \) the opposite algebra for \( U(A) \). For \( c \)-semisimple \( A \), \( W \) is semisimple. For a direct imbedding \( B \) let \( \sigma \) be the natural algebra automorphism with \( \sigma(x) = -x \) and fixing \( x \circ y \) when \( x, y \in A \).

**Proposition 1.** Let \( M \) be an associative trimodule over \( A, K \) a field with \( \text{char}(K) \neq 2 \), and \( A \) \( c \)-semisimple. Then \( M \) is completely reducible over \( K \).

Proof. \( M_U \) is an associative left \( W \)-module by \( x \otimes y \cdot m = x \circ m \circ y \) with \( x, y \in U(A), m \in M_U \). Then \( M_U = e \circ M_U \circ e \oplus M_0 \), \( e \) the unit of \( U(A) \), \( M_0 \) the maximal zero \( W \)-submodule, is a decomposition into \( W \)-submodules. Set \( M := e \circ M_U \circ e \). It may be noted that for the 2nd kind \( e = e_1 + e_2 \) with \( e_1 \in A \circ A, e_2 \in A \circ A \), so that \( \epsilon_i(M) = e_1 \circ M_U \circ e_2 \). Thus \( M = \epsilon_i(M) \oplus \epsilon_i(M_0) \) is a decomposition of \( M \) into subtrimodules. Consider now \( \epsilon_i(M_0) \) as a left- and a right-module over the semisimple algebra \( A \circ A \) over which it decomposes into irreducible submodules [6], and equally for \( A \circ A \oplus A \circ A \) in case of the 2nd kind. Hence \( \epsilon_i(M_0) \) is completely reducible over \( A \).

Set \( G := \{\text{Id, } \sigma\} \) for the 1st kind. Then \( G(M) \subset M \). If \( M \neq \{0\}, M \) decomposes by Lemma 2 or [5, p. 117] into minimal \( G \)-invariant submodules \( P_i \neq \{0\} \) over \( W, k = 1, \ldots, t \). Therefore \( \epsilon_i(P_k) \subset P_k \). Hence \( N_k := \epsilon_i(P_k) \) is an irreducible subtrimodule. Thus \( M \) is a completely reducible \( A \)-trimodule.
The proposition says equivalently that under its conditions on $A$ any short exact sequence of associative $A$-trimodules of the same kind, the connecting mappings being trimodule homomorphisms, splits by a trimodule homomorphism.

An extension of the associative triple $A$ by $M$ is a short exact sequence of associative triples $M, E, A$ of the same kind $\{0\} \rightarrow M \rightarrow^e E \rightarrow^p A \rightarrow \{0\}$ with $e, p$ triple homomorphisms. $M$ is in a canonical way a trimodule over $A$. Let $(1)M := \langle MAM \rangle + \langle MAM \rangle + \langle AMM \rangle$. The extension is singular if $(1)M = \langle MMM \rangle = \{0\}$. The extension is split or trivial if there exists a triple homomorphism $q: A \rightarrow E$, with $pq = \text{Id}_A$. The extension $\{0\} \rightarrow M \rightarrow^f E' \rightarrow^q A \rightarrow \{0\}$ is equivalent to the first if there is a triple homomorphism $k: E \rightarrow E'$ so that the diagram

$$
\begin{array}{ccc}
\{0\} & \rightarrow & M \\
& e & \downarrow \\
& E & \rightarrow \\
& k & \downarrow \\
& A & \rightarrow \{0\} \\
& f & \downarrow \\
& E' & \rightarrow \\
& q & \downarrow \\
\end{array}
$$

is commutative. Then $k$ is an isomorphism. Let $J'$ be the ideal of $U(E)$ defined as $J$, substituting $e(M)$, $E$ for $M$ and $A + M$. Set $U(E)': = U(E)/J'$. Design the maps induced by $e$ and $j$ on $U(E)'$ equally. For a singular extension there is a commutative diagram:

$$
\begin{array}{ccc}
\{0\} & \rightarrow & M \\
& e & \downarrow \\
& E & \rightarrow \\
& p & \downarrow \\
& A & \rightarrow \{0\} \\
& e' & \downarrow \\
& U(E)' & \rightarrow \\
& p' & \downarrow \\
& U(A) & \rightarrow \{0\} \\
\end{array}
$$

e', p'$ algebra homomorphisms invariant relative to $e$ and respectively for $j$. Note that the second row is also exact (cf. [2, pp. 158–159]).

A factor set $f$ of $A$ in $M$ is, as usual, a trilinear mapping $f: A \times A \times A \rightarrow M$ so that the $K$-module direct sum $A + M$ with $[xyz] := \langle xyz \rangle + f(x, y, z)$, $(1)M := \{0\}$, $\langle MMM \rangle := \{0\}$, and the trimodule structure of $M$, is an associative triple of the same kind. $f$ is trivial if there is $s: A \rightarrow M$, $s \cdot$ linear with $f(x, y, z) := s(\langle xyz \rangle) - s(x)yz - xs(y)z - sxs(z)$.

Let $A$ be $K$-projective in the usual sense. Then any extension of the triple $A$ is split for the $K$-module $A$. Similarly as for algebras, the classes of equivalent singular extensions of $A$ by $M$ constitute a $K$-module isomorphic to the $K$-module of factor sets, the trivial factor sets corresponding to the classes of the split extensions. One then obtains, arguing as [2, pp. 159–160] for Lie triples,

**Theorem 2.** If $A$ and the trimodule $M$ over $A$ are associative of the same kind, $A$ $K$-projective, and $\frac{1}{2} \in K$ in case of the 2nd kind, then there is a module isomorphism $\tilde{H}^2(A, M) \cong E(A, M)/E_r(A, M)$, $E(A, M)$ the $K$-module of the classes of equivalent singular extensions of $A$ with kernel $M$, $E_r(A, M)$ the submodule of the classes of split extensions. \[\Box\]
If $A^{(1)} = \langle AAA \rangle$, $A^{(n+1)} = (A^{(n)})^{(1)}$, $n \in \mathbb{N}$, then $A$ is solvable when $A^{(m)} = \{0\}$ for some $m$. Set for an ideal $I$ in $A$, $(n+1)I := (1)^{(n)}I$. $I$ is S-solvable if for one $m$, $(m)I = \{0\}$. Because $R := \text{Rad}(A) \subset \text{Rad}(U(A))$, $R$ is solvable if $K$ is a field. We have for arbitrary $K$

**Lemma 3.** Let $I$ be an ideal of $A$. Then:

(a) $I$ is solvable exactly if it is S-solvable.

(b) When $I$ is solvable, $I \neq \{0\}$, then $(1)I \subsetneq I$.

**Proof.** (a) From $B^{(1)} \subset (1)B$ we get $B^{(n)} \subset (n)B$ for any $n \in \mathbb{N}$. Further one verifies $(3)B \subset B^{(1)}$. Thus $(3n)B \subset B^{(n)}$. Hence we get (a). (a) $\Rightarrow$ (b).

The proof of the Wedderburn principal theorem reduces by Lemma 3 to the singular case. For associative triples of the 2nd kind the statement is implied by [1]. On behalf of Corollary 1 it follows:

**Corollary 3.** Let $A$ be a finite dimensional associative triple system over a field $K$, $R$ the radical, $A/R$ separable, and $\text{char}(K) \neq 2$. Then $A$ decomposes $A = B \oplus R$, $B$ a subtriple of $A$, $B \cong A/R$.

**References**