

A CLASS OF REGULAR MATRICES

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ABSTRACT. Let m be the space of real, bounded sequences $x = \{x_k\}$ with the sup norm, and let $A = (a_{n,k})$ be a regular (i.e., Toeplitz) matrix. We consider the following two possible conditions for A : (1) $\sum_{k=1}^{\infty} |a_{n,k}| \rightarrow 1$ as $n \rightarrow \infty$, (2) $\sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}| \rightarrow 0$ as $n \rightarrow \infty$. G. Das [J. London Math. Soc. (2) 7 (1974), 501-507] proved that if a regular matrix A satisfies both (1) and (2) then (3) $\overline{\lim}_{n \rightarrow \infty} (Ax)_n < q(x)$ for all $x \in m$, where $q(x) = \inf_{n,p} \overline{\lim}_{k \rightarrow \infty} p^{-1} \sum_{i=1}^p x_{n_i+k}$. Das used "Banach limits" and Hahn-Banach techniques, and stated that he thought it would be "difficult to establish the result . . . by direct method". In the present paper an elementary proof of the result is given, and it is shown also that the converse holds, i.e., for a regular A , (3) implies (1) and (2). Hence (3) completely characterizes the class of regular matrices satisfying (1) and (2).

1. We write m for the space of real, bounded sequences $x = \{x_k\}$ with the sup norm, and we say that an infinite, real matrix $A = (a_{n,k})$ is regular iff $\lim_{n \rightarrow \infty} (Ax)_n = c$ when $\lim_{n \rightarrow \infty} x_n = c$. It is known (see [2, p. 502]) that A is regular iff (a) $\sup_n \sum_{k=1}^{\infty} |a_{n,k}| < \infty$, (b) $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{n,k} = 1$, (c) $\lim_{n \rightarrow \infty} a_{n,k} = 0 \forall k$. From (a) it is clear that $Ax \in m$ for all $x \in m$.

We shall be concerned with two classes of regular matrices: the one consists of those regular matrices $A = (a_{n,k})$ for which $(|a_{n,k}|)$ is regular. From the conditions above this happens iff

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{n,k}| = 1.$$

The other class consists of those regular matrices A whose action on an $x \in m$ is asymptotically independent of a "shift" in x , i.e., which satisfy $\lim_{n \rightarrow \infty} (A(x - Dx))_n = 0$ for all $x \in m$, where, if $x = \{x_k\}$, $Dx = \{x_{k+1}\}$. By Lemma 3 below this happens iff

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{n,k} - a_{n,k+1}| = 0.$$

We shall prove

THEOREM 1. *For a regular matrix A the following are equivalent:*

(i) *A satisfies (1) and (2);*

$$(3)(ii) \quad \overline{\lim}_{n \rightarrow \infty} (Ax)_n \leq q(x) \quad \text{for all } x \in m,$$

Presented to the Society, August 23, 1974; received by the editors January 28, 1975.

AMS (MOS) subject classifications (1970). Primary 40C05, 40H05; Secondary 46B15.

Key words and phrases. Regular matrices, Toeplitz matrices, Banach limits, almost convergence.

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where

$$q(x) = \inf_{n,p} \overline{\lim}_{k \rightarrow \infty} p^{-1} \sum_{i=1}^p x_{n_i+k}.$$

The quantity $q(x)$ arises in connection with Banach limits [2, p. 501]: the Banach limits on m are precisely the linear functionals f on m satisfying $f(x) \leq q(x)$ [2, Theorem 1]. From (3) we thus have that every $x \in m$ which is almost convergent (i.e., for which $-q(-x) = q(x)$) is summed by A to $q(x)$, which in turn is the common value of all the Banach limits.

G. Das [2, Corollary to Theorem 2] proved that (i) implies (ii) by using Banach limits and Hahn-Banach techniques. He stated that he thought it would be “difficult to establish the result . . . by direct method”. Our proof of Theorem 1 below uses only elementary methods.

2. Lemmas. Lemma 1 and its proof are substantially due to Lorentz [4, pp. 170–171].

LEMMA 1. *If $x \in m$ then $q^*(x) = \overline{\lim}_{K \rightarrow \infty} K^{-1} \sum_{k=1}^K x_{k+n}$ is independent of n and satisfies $q^*(x) \leq q(x)$ uniformly in n , i.e., given $\epsilon > 0$ there exists k' independent of n such that*

$$(4) \quad K^{-1} \sum_{k=1}^K x_{k+n} < q(x) + \epsilon \quad \text{for all } K \geq k' \text{ and all } n \geq 0.$$

PROOF. Taking two consequent values of n in the expression for $q^*(x)$ and subtracting shows at once that $q^*(x)$ is independent of n . Now, writing $x_k = (x_k - b) + b$, where $b = \inf x_k$, it is sufficient for the rest to take $x_k \geq 0$ for all k . By the definition of $q(x)$ we see that given $\epsilon > 0$ there exist n_i, p, k_ϵ ($n_i \leq n_{i+1}$) such that $p^{-1} \sum_{i=1}^p x_{n_i+k+n} < q(x) + (\epsilon/2)$ for all $k \geq k_\epsilon$, and all $n \geq 0$. Write $k^* = \max(k_\epsilon, n_p)$. For $K > k^*$ consider

$$S = K^{-1} \sum_{k=1}^K p^{-1} \sum_{i=1}^p x_{n_i+k+n}.$$

Splitting the outer sum into sums over the ranges $1 \leq k \leq k^*$ and $k^* + 1 \leq k \leq K$, we see that the first term obtained is $\leq (k^* \|x\|)/K < \epsilon/4$ for $K > \text{some } k_1$, where k_1 is independent of $n \geq 0$, and the second is $\leq q(x) + (\epsilon/2)$.

Rewriting S as $(Kp)^{-1} \sum_{i=1}^p \sum_{k=1}^K x_{n_i+k+n}$ and then splitting the inner sum into sums having respective ranges $1 \leq k \leq n_p - n_i, n_p - n_i + 1 \leq k \leq K - n_i + n_1, K - n_i + n_1 + 1 \leq k \leq K$, we see that with $S = A + B + C$,

$$0 \leq A + C \leq (2n_p p \|x\|/p) K^{-1} < \epsilon/8 \quad \text{for all } K > \text{some } k_2$$

where k_2 is independent of $n \geq 0$, and

$$\begin{aligned} 0 < B &= K^{-1} \sum_{r=n_p+1}^{n_1+K} x_{r+n} = K^{-1} \left\{ \sum_{r=1}^K - \sum_{r=1}^{n_p} + \sum_{r=K+1}^{K+n_1} \right\} x_{r+n}, \\ &= T - U + V, \end{aligned}$$

say. Clearly, U and V are each nonnegative and $\leq (\|x\| n_p)/K < \epsilon/16$ for all $K > \text{some } k_3$, where k_3 is independent of $n \geq 0$; hence

$$0 \leq T = B + U - V = S - A - C + U - V < q(x) + (3\epsilon/4) + (\epsilon/8) + (\epsilon/8) = q(x) + \epsilon$$

for all $K > k'$, where $k' = \max(k^*, k_1, k_2, k_3)$, k' being independent of $n \geq 0$. This gives (4).

LEMMA 2. For a regular matrix A the following are equivalent:

- (i) A satisfies (1);
- (ii) $\overline{\lim}_{n \rightarrow \infty} (Ax)_n \leq \overline{\lim}_{n \rightarrow \infty} x_n$ for all $x \in m$.

This is given in [2, p. 503] (τ^+ in (2.4) should read τ). An elementary proof is given in [1, pp. 150–152].

LEMMA 3. For a regular matrix A , (2) is true iff

$$(5) \quad \lim_{n \rightarrow \infty} (A(x - Dx))_n = 0 \quad \text{for all } x \in m,$$

where, if $x = \{x_k\}$, $Dx = \{x_{k+1}\}$.

This is given in [2, p. 502]. If we write (5) as $\lim_{n \rightarrow \infty} ((A - AD)x)_n = 0$ for all $x \in m$, so that A and AD are “absolutely equivalent”, the result is covered by the elementary proof in [1, pp. 105–107].

3. Proof of Theorem 1: (i) \Rightarrow (ii). As in the proof of Lemma 1, it is sufficient to take $x_k \geq 0$ for all k . Now by the convergence of the series in (1),

$$(6) \quad (Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k = \sum_{t=0}^{\infty} \sum_{k=tk'+1}^{tk'+k'} a_{n,k} x_k \quad \text{for all } x \in m,$$

where, for given $\epsilon > 0$, k' satisfies (4). Now by writing $x_k = s_k - s_{k-1}$, where

$$s_k = \begin{cases} \sum_{k=v}^k x_r, & k \geq v, \\ 0, & k < v, \end{cases}$$

we have (cf. [3, line 1, p. 437])

$$(7) \quad \sum_{k=v}^w a_{n,k} x_k = \sum_{k=v}^{w-1} s_k (a_{n,k} - a_{n,k+1}) + s_w a_{n,w} = E + F,$$

say. We apply this with $v = tk' + 1$, $w = tk' + k'$; since

$$0 \leq s_k \leq s_w = \sum_{p=1}^{k'} x_{p+tk'} < k'(q(x) + \epsilon) \quad \text{for all } t \geq 0,$$

we obtain

$$(8) \quad E + F \leq k'(q(x) + \epsilon) \left(\sum_{k=tk'+1}^{tk'+k'} |a_{n,k} - a_{n,k+1}| + |a_{n,tk'+k'}| \right).$$

Now

$$\begin{aligned} k'|a_{n,tk'+k'}| &= \sum_{s=2}^{k'+1} |a_{n,tk'+k'}| \\ &= \sum_{s=2}^{k'+1} \left(|a_{n,tk'+s-1}| + \sum_{r=tk'+s}^{tk'+k'} (|a_{n,r}| - |a_{n,r-1}|) \right). \end{aligned}$$

Writing $r = tk' + s - 1$ in the first sum, and majorizing $|a_{n,r}| - |a_{n,r-1}|$ by $|a_{n,r} - a_{n,r-1}|$ and then replacing $r = tk' + s$ by $r = tk' + 2$ in the second sum, we obtain, by (6)–(8),

$$(Ax)_n \leq (q(x) + \varepsilon) \left(\sum_{r=1}^{\infty} |a_{n,r}| + 2k' \sum_{r=2}^{\infty} |a_{n,r} - a_{n,r-1}| \right)$$

for all $n \geq 0$, k' being independent of n . Hence

$$\overline{\lim}_{n \rightarrow \infty} (Ax)_n \leq (q(x) + \varepsilon)(1 + 2k' \cdot 0) = q(x) + \varepsilon.$$

Since ε is arbitrarily small, we obtain (3).

(ii) \Rightarrow (i). (1) follows at once by Lemma 2 and the fact that (by taking $n_i = p = 1$) $q(x) \leq \overline{\lim}_{n \rightarrow \infty} x_n$. Now as in [2, p. 502],

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} (A(x - Dx))_n &\leq q(x - Dx) \leq \overline{\lim}_{k \rightarrow \infty} p^{-1} \sum_{i=1}^p (x_{i+k} - x_{i+k+1}) \quad \text{for all } p, \\ &\leq \overline{\lim}_{k \rightarrow \infty} 2\|x\|/p. \end{aligned}$$

Since p is arbitrarily large, $\overline{\lim}_{n \rightarrow \infty} (A(x - Dx))_n \leq 0$. Replacing $x - Dx$ by $Dx - x$ we get $\overline{\lim}_{n \rightarrow \infty} (A(Dx - x))_n \leq 0$, i.e. $\overline{\lim}_{n \rightarrow \infty} (A(x - Dx))_n \geq 0$. This gives (5), and hence (2), by Lemma 3.

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