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A CLASS OF REGULAR MATRICES

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ABSTRACT. Let m be the space of real, bounded sequences $x = \{x_k\}$ with the sup norm, and let $A = (a_{n,k})$ be a regular (i.e., Toeplitz) matrix. We consider the following two possible conditions for A: (1) $\sum_{k=1}^{\infty}|a_{n,k}| \to 1$ as $n \to \infty$, (2) $\sum_{k=1}^{\infty}|a_{n,k} - a_{n,k+1}| \to 0$ as $n \to \infty$. G. Das [J. London Math. Soc. (2) 7 (1974), 501–507] proved that if a regular matrix A satisfies both (1) and (2) then (3) $\overline{\lim}_{n\to\infty}(Ax)_n \leqslant q(x)$ for all $x \in m$, where $q(x) = \inf_{n_i,p}\overline{\lim}_{k\to\infty} p^{-1}\sum_{i=1}^{p} x_{n_i+k}$. Das used "Banach limits" and Hahn-Banach techniques, and stated that he thought it would be "difficult to establish the result . . . by direct method". In the present paper an elementary proof of the result is given, and it is shown also that the converse holds, i.e., for a regular A, (3) implies (1) and (2). Hence (3) completely characterizes the class of regular matrices satisfying (1) and (2).

1. We write m for the space of real, bounded sequences $x = \{x_k\}$ with the sup norm, and we say that an infinite, real matrix $A = (a_{n,k})$ is regular iff $\lim_{n\to\infty}(Ax)_n = c$ when $\lim_{n\to\infty}x_n = c$. It is known (see [2, p. 502]) that A is regular iff (a) $\sup_n\sum_{k=1}^{\infty}|a_{n,k}| < \infty$, (b) $\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{n,k} = 1$, (c) $\lim_{n\to\infty}a_{n,k} = 0\ \forall k$. From (a) it is clear that $Ax \in m$ for all $x \in m$.

We shall be concerned with two classes of regular matrices: the one consists of those regular matrices $A = (a_{n,k})$ for which $(|a_{n,k}|)$ is regular. From the conditions above this happens iff

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}|a_{n,k}| = 1. \tag{1}$$

The other class consists of those regular matrices A whose action on an $x \in m$ is asymptotically independent of a "shift" in x, i.e., which satisfy $\lim_{n\to\infty}(A(x - Dx))_n = 0$ for all $x \in m$, where, if $x = \{x_k\}$, $Dx = \{x_{k+1}\}$. By Lemma 3 below this happens iff

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}|a_{n,k} - a_{n,k+1}| = 0. \tag{2}$$

We shall prove

THEOREM 1. *For a regular matrix A the following are equivalent*:
(i) *A satisfies* (1) *and* (2);

$$\overline{\lim_{n\to\infty}}\,(Ax)_n \leqslant q(x) \quad \textit{for all } x \in m, \tag{3}$$ (ii)

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where

$$q(x) = \inf_{n_i, p} \overline{\lim_{k \to \infty}} \, p^{-1} \sum_{i=1}^{p} x_{n_i + k}.$$

The quantity $q(x)$ arises in connection with Banach limits [2, p. 501]: the Banach limits on m are precisely the linear functionals f on m satisfying $f(x) \leqslant q(x)$ [2, Theorem 1]. From (3) we thus have that every $x \in m$ which is almost convergent (i.e., for which $-q(-x) = q(x)$) is summed by A to $q(x)$, which in turn is the common value of all the Banach limits.

G. Das [2, Corollary to Theorem 2] proved that (i) implies (ii) by using Banach limits and Hahn-Banach techniques. He stated that he thought it would be "difficult to establish the result . . . by direct method". Our proof of Theorem 1 below uses only elementary methods.

2. **Lemmas.** Lemma 1 and its proof are substantially due to Lorentz [4, pp. 170–171].

LEMMA 1. *If $x \in m$ then $q^*(x) = \overline{\lim}_{K \to \infty} K^{-1} \sum_{k=1}^{K} x_{k+n}$ is independent of n and satisfies $q^*(x) \leqslant q(x)$ uniformly in n, i.e., given $\varepsilon > 0$ there exists k' independent of n such that*

$$K^{-1} \sum_{k=1}^{K} x_{k+n} < q(x) + \varepsilon \quad \textit{for all } K \geqslant k' \textit{ and all } n \geqslant 0. \tag{4}$$

PROOF. Taking two consequent values of n in the expression for $q^*(x)$ and subtracting shows at once that $q^*(x)$ is independent of n. Now, writing $x_k = (x_k - b) + b$, where $b = \inf x_k$, it is sufficient for the rest to take $x_k \geqslant 0$ for all k. By the definition of $q(x)$ we see that given $\varepsilon > 0$ there exist n_i, p, k_ε $(n_i \leqslant n_{i+1})$ such that $p^{-1} \sum_{i=1}^{p} x_{n_i + k + n} < q(x) + (\varepsilon/2)$ for all $k \geqslant k_\varepsilon$, and all $n \geqslant 0$. Write $k^* = \max(k_\varepsilon, n_p)$. For $K > k^*$ consider

$$S = K^{-1} \sum_{k=1}^{K} p^{-1} \sum_{i=1}^{p} x_{n_i + k + n}.$$

Splitting the outer sum into sums over the ranges $1 \leqslant k \leqslant k^*$ and $k^* + 1 \leqslant k \leqslant K$, we see that the first term obtained is $\leqslant (k^* \|x\|)/K < \varepsilon/4$ for $K >$ some k_1, where k_1 is independent of $n \geqslant 0$, and the second is $\leqslant q(x) + (\varepsilon/2)$.

Rewriting S as $(Kp)^{-1} \sum_{i=1}^{p} \sum_{k=1}^{K} x_{n_i + k + n}$ and then splitting the inner sum into sums having respective ranges $1 \leqslant k \leqslant n_p - n_i$, $n_p - n_i + 1 \leqslant k \leqslant K - n_i + n_1$, $K - n_i + n_1 + 1 \leqslant k \leqslant K$, we see that with $S = A + B + C$,

$$0 \leqslant A + C \leqslant (2 n_p p \|x\|/p) K^{-1} < \varepsilon/8 \quad \text{for all } K > \text{some } k_2$$

where k_2 is independent of $n \geqslant 0$, and

$$0 \leqslant B = K^{-1} \sum_{r=n_p+1}^{n_1+K} x_{r+n} = K^{-1} \left\{ \sum_{r=1}^{K} - \sum_{r=1}^{n_p} + \sum_{r=K+1}^{K+n_1} \right\} x_{r+n},$$

$$= T - U + V,$$

say. Clearly, U and V are each nonnegative and $\leqslant (\|x\| n_p)/K < \varepsilon/16$ for all $K >$ some k_3, where k_3 is independent of $n \geqslant 0$; hence

$$0 \leqslant T = B + U - V = S - A - C + U - V$$
$$< q(x) + (3\varepsilon/4) + (\varepsilon/8) + (\varepsilon/8) = q(x) + \varepsilon$$

for all $K > k'$, where $k' = \max(k^*, k_1, k_2, k_3)$, k' being independent of $n \geqslant 0$. This gives (4).

LEMMA 2. *For a regular matrix A the following are equivalent*:
(i) *A satisfies* (1);
(ii) $\overline{\lim}_{n\to\infty}(Ax)_n \leqslant \overline{\lim}_{n\to\infty} x_n$ *for all* $x \in m$.

This is given in [2, p. 503] (τ^+ in (2.4) should read τ). An elementary proof is given in [1, pp. 150–152].

LEMMA 3. *For a regular matrix A*, (2) *is true iff*

$$\lim_{n\to\infty} (A(x - Dx))_n = 0 \quad \textit{for all } x \in m, \tag{5}$$

where, if $x = \{x_k\}$, $Dx = \{x_{k+1}\}$.

This is given in [2, p. 502]. If we write (5) as $\lim_{n\to\infty}((A - AD)x)_n = 0$ for all $x \in m$, so that A and AD are "absolutely equivalent", the result is covered by the elementary proof in [1, pp. 105–107].

3. **Proof of Theorem 1:** (i) $\Rightarrow$ (ii). As in the proof of Lemma 1, it is sufficient to take $x_k \geqslant 0$ for all k. Now by the convergence of the series in (1),

$$(Ax)_n = \sum_{k=1}^{\infty} a_{n,k}x_k = \sum_{t=0}^{\infty} \sum_{k=tk'+1}^{tk'+k'} a_{n,k}x_k \quad \text{for all } x \in m, \tag{6}$$

where, for given $\varepsilon > 0$, k' satisfies (4). Now by writing $x_k = s_k - s_{k-1}$, where

$$s_k = \begin{cases} \sum_{k=v}^{k} x_r, & k \geqslant v, \\ 0, & k < v, \end{cases}$$

we have (cf. [3, line 1, p. 437])

$$\sum_{k=v}^{w} a_{n,k}x_k = \sum_{k=v}^{w-1} s_k(a_{n,k} - a_{n,k+1}) + s_w a_{n,w} = E + F, \tag{7}$$

say. We apply this with $v = tk' + 1$, $w = tk' + k'$; since

$$0 \leqslant s_k \leqslant s_w = \sum_{p=1}^{k'} x_{p+tk'} < k'(q(x) + \varepsilon) \quad \text{for all } t \geqslant 0,$$

we obtain

$$E + F \leqslant k'(q(x) + \varepsilon)\left(\sum_{k=tk'+1}^{tk'+k'} |a_{n,k} - a_{n,k+1}| + |a_{n,tk'+k'}|\right). \tag{8}$$

Now

$$k'|a_{n,tk'+k'}| = \sum_{s=2}^{k'+1} |a_{n,tk'+k'}|$$
$$= \sum_{s=2}^{k'+1} \left(|a_{n,tk'+s-1}| + \sum_{r=tk'+s}^{tk'+k'} (|a_{n,r}| - |a_{n,r-1}|)\right).$$

Writing $r = tk' + s - 1$ in the first sum, and majorizing $|a_{n,r}| - |a_{n,r-1}|$ by $|a_{n,r} - a_{n,r-1}|$ and then replacing $r = tk' + s$ by $r = tk' + 2$ in the second sum, we obtain, by (6)–(8),

$$(Ax)_n \leqslant (q(x) + \varepsilon)\left(\sum_{r=1}^{\infty} |a_{n,r}| + 2k' \sum_{r=2}^{\infty} |a_{n,r} - a_{n,r-1}|\right)$$

for all $n \geqslant 0$, k' being independent of n. Hence

$$\overline{\lim_{n\to\infty}} (Ax)_n \leqslant (q(x) + \varepsilon)(1 + 2k' \cdot 0) = q(x) + \varepsilon.$$

Since ε is arbitrarily small, we obtain (3).

(ii) $\Rightarrow$ (i). (1) follows at once by Lemma 2 and the fact that (by taking $n_i = p = 1$) $q(x) \leqslant \overline{\lim}_{n\to\infty} x_n$. Now as in [2, p. 502],

$$\overline{\lim_{n\to\infty}} (A(x - Dx))_n \leqslant q(x - Dx) \leqslant \overline{\lim_{k\to\infty}}\, p^{-1} \sum_{i=1}^{p} (x_{i+k} - x_{i+k+1}) \quad \text{for all } p,$$

$$\leqslant \overline{\lim_{k\to\infty}}\, 2\|x\|/p.$$

Since p is arbitrarily large, $\overline{\lim}_{n\to\infty}(A(x - Dx))_n \leqslant 0$. Replacing $x - Dx$ by $Dx - x$ we get $\overline{\lim}_{n\to\infty}(A(Dx - x))_n \leqslant 0$, i.e. $\underline{\lim}_{n\to\infty}(A(x - Dx))_n \geqslant 0$. This gives (5), and hence (2), by Lemma 3.

References

1. R. G. Cooke, *Infinite matrices and sequence spaces*, Dover, New York, 1955.

2. G. Das, *Banach and other limits*, J. London Math. Soc. (2) **7** (1974), 501–507. MR **49** #924.

3. G. L. Isaacs, *An iteration formula for fractional differences*, Proc. London Math. Soc. (3) **13** (1963), 430–460. MR **27** #5061.

4. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190. MR **10**, 367.

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