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PULL-BACKS OF FORMATION PROJECTORS

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ABSTRACT. Pull-backs for the projectors associated with a formation of finite solvable groups are constructed. An application is given to the construction of certain interesting formations.

1. **Introduction.** In this note, we show that "pull-backs" exist for the $\mathfrak{F}$-projectors associated with a saturated formation $\mathfrak{F}$ of finite solvable groups. This result enables us to construct in a uniform manner several interesting classes of formations.

Our notation and terminology are fairly standard, and can be understood by referring to [4] or [6]. The latter source contains the basic material on formations needed here. Note that we use the more current term *$\mathfrak{F}$-projector* for what were formerly called *$\mathfrak{F}$-covering subgroups.*

2. **Preliminaries.** The constructions of §§3 and 4 depend on an important theorem of B. Huppert [7].

THEOREM 2.1. *Suppose that $\mathfrak{F}$ is a saturated formation, and that F is an $\mathfrak{F}$-projector of a solvable group G. Then for all normal subgroups M and N of G,*

$$F \cap MN = (F \cap M)(F \cap N).$$

We shall need the next result in the proof of Theorem 4.2.

THEOREM 2.2 [5]. *Suppose that C is a subgroup of a solvable group G. If C centralizes each p-chief factor of $G/\Phi(G)$, then C centralizes each p-chief factor of G below $\Phi(G)$.*

The following proposition is common folklore, although not perhaps in this form. We denote by $\mathfrak{F}_1\mathfrak{F}_2$ the class of all groups which are extensions of groups in $\mathfrak{F}_1$ by groups in $\mathfrak{F}_2$.

PROPOSITION 2.3. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be saturated formations, and suppose that the order of each group in $\mathfrak{F}_1$ is coprime to the order of each group in $\mathfrak{F}_2$. Then $\mathfrak{F}_1\mathfrak{F}_2$ is a formation. If F_1 is an $\mathfrak{F}_1$-projector of G, and F_2 is an $\mathfrak{F}_2$-projector of $N_G(F_1)$, then F_1F_2 is an $\mathfrak{F}_1\mathfrak{F}_2$-projector of G.*

PROOF. The class $\mathfrak{F}_1\mathfrak{F}_2$ is clearly quotient closed. Suppose that G/M and G/N belong to $\mathfrak{F}_1\mathfrak{F}_2$. Then for some normal subgroups R, S, we have

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G/R, $G/S \in \mathfrak{F}_2$, while R/M, $S/N \in \mathfrak{F}_1$. Choose a Hall π-subgroup H of G, where π is the set of primes dividing $|R/M|$ or $|S/N|$. Then by the hypothesis on $\mathfrak{F}_1$ and $\mathfrak{F}_2$, $H \leqslant R \cap S$, and $HM = R$, $HN = S$. Thus $H/H \cap M \cong R/M$ and $H/H \cap N \cong S/N$. Since $R \cap S/M \cap N$ is a π-group, $H(M \cap N) = R \cap S$, and so

$$R \cap S/M \cap N \cong H/H \cap M \cap N \in \mathfrak{F}_1.$$

Now since $G/R \cap S \in \mathfrak{F}_2$, we have $G/M \cap N \in \mathfrak{F}_1\mathfrak{F}_2$, as desired.

We consider the second conclusion. Certainly $F_1F_2 \in \mathfrak{F}_1\mathfrak{F}_2$. Now suppose that $F_1F_2 \leqslant V$, and $V/N \in \mathfrak{F}_1\mathfrak{F}_2$. We must verify that $F_1F_2N = V$. Since $V/N \in \mathfrak{F}_1\mathfrak{F}_2$, there is a normal subgroup K of V so that $V/K \in \mathfrak{F}_2$ and $K/N \in \mathfrak{F}_1$. Since $|V/K|$ is coprime to $|F_1|$, $F_1 \leqslant K$, and therefore F_1 is an $\mathfrak{F}_1$-projector of K. Thus $K = F_1N$. Since any two V-conjugates of F_1 are necessarily K-conjugate, $V = K \cdot N_V(F_1)$. Therefore $V/K \cong N_V(F_1) \cap K$. This last quotient belongs to $\mathfrak{F}_2$, and $N_V(F_1)$ contains F_2, an $\mathfrak{F}_2$-projector of $N(F_1)$. It follows that $F_2(N_V(F_1) \cap K) = N_V(F_1)$; now $V = KN_V(F_1) = KF_2 = NF_1F_2$, completing the proof.

3. **Pull-backs for $\mathfrak{F}$-projectors.** We show now that pull-backs exist for the projectors associated with a saturated formation $\mathfrak{F}$.

THEOREM 3.1. *Let $\mathfrak{F}$ be a saturated formation. Let G be a solvable group with normal subgroups M and N, and suppose that $M \cap N = 1$. If an $\mathfrak{F}$-projector of G/M and an $\mathfrak{F}$-projector of G/N have the same image in G/MN, then they are both homomorphic images of some $\mathfrak{F}$-projector of G.*

PROOF. Since pre-images of $\mathfrak{F}$-projectors contain $\mathfrak{F}$-projectors, we may suppose that there are $\mathfrak{F}$-projectors F_1, F_2 of G, such that $F_1MN = F_2MN$. We must find an $\mathfrak{F}$-projector F so that $FM = F_1M$ and $FN \Rightarrow F_2N$. It is routine to see that if G is a minimal counterexample to the theorem, $G = F_1MN$ and M and N are minimal normal subgroups in G, and therefore abelian. A nontrivial intersection $F_1 \cap M$ or $F_1 \cap N$ would be normal in G; therefore F_1 must either contain or have trivial intersection with each of M and N. Suppose that $M \leqslant F_1$. Then $G/N = F_1N/N \in \mathfrak{F}$, so that $F_2N = G$. Thus we may take $F = F_1$ and be finished, in this case. By symmetry, we may suppose that $M \cap F_1 = N \cap F_1 = 1$. Now by the theorem of B. Huppert given above, $F_1 \cap MN = 1$, and we may suppose that similar results hold for F_2. Now let $F = F_1M \cap F_2N$. Then $|F| = |F_1M||F_2M|/|G| = |F_1|$. Now $MF = M(F_1M \cap F_2N) = F_1M \cap G$ by the modular law. This shows that F and F_1 are both complements to M in MF_1, and are therefore conjugate. In particular, F is an $\mathfrak{F}$-projector of G. Since we can show in identical fashion that $NF = NF_2$, our proof is complete.

4. **Constructing formations by comparing projectors.** It is easy to see that the class of groups for which $\mathfrak{F}_1$-projectors belong to $\mathfrak{F}_2$ is a formation; A-groups, and groups with Abelian Carter subgroups are two familiar formations of this type. As an application of the results above, we offer a more delicate construction.

THEOREM 4.1. *Suppose that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are saturated formations; then the class* *of groups in which $\mathfrak{F}_1$-projectors are subgroups of $\mathfrak{F}_2$-projectors is a formation. In particular, the class of groups for which $\mathfrak{F}_1$- and $\mathfrak{F}_2$-projectors coincide is a formation.*

PROOF. The second statement is a direct consequence of the first. Since projectors are carried onto projectors by epimorphisms, these classes are quotient closed. Now suppose that M and N are normal in G, $M \cap N = 1$, and in both G/M and G/N, $\mathfrak{F}$-projectors are subgroups of $\mathfrak{F}_1$-projectors. Then for an $\mathfrak{F}_2$-projector F_2 of G, F_2M/M and F_2N/N are $\mathfrak{F}_2$-projectors of G/M and G/N. Furthermore, by lifting back from G/MN into G/M and G/N, and then applying Theorem 3.1, we may assume that for some $\mathfrak{F}_1$-projector F_1, we have $F_1M \leqslant F_2M$ and $F_1N \leqslant F_2N$. Applying 2.1, we see that

$$F_1 = F_1(M \cap N) = F_1M \cap F_1N \leqslant F_2M \cap F_2M = F_2(M \cap N) = F_2;$$

our proof is complete.

Several important formations arise in the manner of Theorem 4.1. For instance, groups in which Carter subgroups (i.e. nilpotent projectors) are Hall subgroups have arisen in investigations by R. Carter, J. G. Thompson and others, of splitting and nilpotency length (see [1], [8]; also [6, p. 743]). J. B. Derr [2] and A. Fattahi [3] have studied a generalization of Sylow tower groups of this type, in which the normalizers of Sylow p-subgroups (projectors for p-by-p' groups) are Hall π-subgroups (projectors for π-groups).

The formations we have constructed are usually not saturated. For example, let G be the nonnilpotent extension of the quaternion group Q_8 by a cyclic group C_3. Then in $G/\Phi(G)$, the Carter subgroup is a Sylow subgroup, but this is false in G. For many classes which are more or less arithmetically defined, we can prove saturation.

THEOREM 4.2. *Let $\pi_1, \ldots, \pi_k$ be disjoint sets of primes, and set $\pi = \bigcup \pi_i$. Let $\mathfrak{S}_{\pi_i}$ denote the formation of solvable π_i-groups. Then the class of groups $\mathfrak{F}$ in which an $\mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_n}$-projector is an $\mathfrak{S}_\pi$-projector forms a saturated formation.*

PROOF. In view of 4.1, only saturation needs proof. Suppose that M is a minimal normal of G, with $M \leqslant \Phi(G)$. Then M is a p-group for some prime p. If $p \notin \pi$, then G/M and G have isomorphic Hall π-subgroups, so that $G \in \mathfrak{F}$ if $G/M \in \mathfrak{F}$. Now suppose that $p \in \pi_i$. Then a Hall π-subgroup of G contains M, so we need to see that an $\mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_n}$-projector does also. In view of 2.3, this is no difficulty if $p \in \pi_1$. If $i > 1$, the hypothesis on G/M and 2.3 show that for $r \in \pi_1 \cup \cdots \cup \pi_{i-1}$, a Sylow r-subgroup centralizes each p-factor of G/M. Now by 2.2, this is also true in G, so M is centralized by a Hall $\pi_1 \cup \cdots \cup \pi_{i-1}$-subgroup of G, and therefore M is contained in an $\mathfrak{S}_{\pi_1} \cdots \mathfrak{S}_{\pi_n}$-projector, as desired. This completes the proof of the theorem.

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