NEW PROOF OF A DENSITY THEOREM
FOR THE BOUNDARY OF A CLOSED SET

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ABSTRACT. From Browder [1] the following theorem is known: Let \( F \) be a closed subset of the Banach space \( E \); then the set \( R \) of points \( x \in \partial F \), such that \( F \cap C = \{x\} \) for at least one convex \( C \) with nonempty interior, is dense in \( \partial F \). A proof of this will be given by means of a theorem of Martin [4] on ordinary differential equations.

Proofs of the just quoted result have been given by Browder [1], [2], Danes [3], and Phelps [5]. A completely different proof runs as follows: Assume the statement of Browder’s theorem to be false. Then there exists a point \( p \in \partial F \) and an open, convex neighbourhood \( U \) of \( p \) such that

\[
R \cap \partial F \cap U = \emptyset.
\]

Now, for every function \( f: \mathbb{R} \to E \) the formula

\[
\lim_{h \to 0^+} \frac{1}{h} |F, x + hf(x)| = 0 \quad (x \in \partial F \cap U)
\]

is valid (see below; \(|F, y|\) denotes the distance from \( F \) to the point \( y \)). Choose \( q \in U \setminus F \) and define

\[
f(x) = q - p \quad (x \in U).
\]

Then the unique solution of the initial value problem

\[
u(0) = p, \quad u'(t) = f(u(t)) \quad (0 \leq t \leq 1)
\]

is \( u(t) = (1 - t)p + tq \), but since \( f \) is Lipschitz-continuous and (2) holds, \( u(t) \) must remain in \( F \) by Theorem 4 of Martin [4]. This yields \( q = u(1) \in F \), leading to a contradiction.

To prove (2), fix \( x \in \partial F \cap U \) and let \( \epsilon, h_0 > 0 \). Then

\[
C_{\epsilon, h_0} = \{x + hf(x) + hes \mid 0 < h < h_0, \|s\| < 1\}
\]

is a convex set with nonempty interior. By (1), \( x \not\in R \), and so there is some \( y \in F \cap C_{\epsilon, h_0}, y \neq x \), i.e.

\[
y = x + hf(x) + hes, \quad \text{where } 0 < h < h_0, \|s\| < 1.
\]

Hence

\[
|F, x + hf(x)| \leq \|x + hf(x) - y\| = he\|s\| < he,
\]

yielding
\( (1/h)|F, x + hf(x)| \leq \varepsilon \) for some \( h \in (0, h_0] \).

Since \( \varepsilon \) and \( h_0 \) have been chosen arbitrarily, (2) is established.

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References


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