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A NOTE ON THE ESSENTIAL SELFADJOINTNESS OF CLASSICAL CONSTANTS OF MOTION

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ABSTRACT. It is shown how the results by Chernoff [1] and also Rauch and Taylor [2] on the essential selfadjointness of powers of generators of hyperbolic mixed problems can be combined with results by Poulsen [3] to give essential selfadjointness of symmetric operators commuting with the hyperbolic problem, as specified below.

Let M be a complete Riemannian manifold with volume element dv and ξ a hermitian bundle over M with $\langle\cdot,\cdot\rangle_x$ the inner product in the fiber over x. As in [1] we consider a first-order symmetric hyperbolic operator L on $C^\infty(\xi)$, all C^∞ sections of ξ, of the form

$$L = \sum_{i=1}^{n} A^i(x)\frac{\partial}{\partial x_i} + B(x).$$

Under the assumption that the local velocity of propagation of solutions to $\partial u/\partial t = Lu$ is sufficiently low, we get [1] that this has unique global solutions given initial data in $C_0^\infty(\xi)$, all C^∞ sections with compact support. Hence we infer the existence of a continuous unitary one-parameter group $V(t)$ in $H = L^2(\xi)$ such that

$$(d/dt)V(t)u = VL(t)u = V(t)Lu, \tag{1}$$

$$V(t)u \in C_0^\infty(\xi) \tag{2}$$

for all $u \in C_0^\infty(\xi)$. In particular, the generator of $V(t)$ contains iL.

LEMMA 1. *Let U be a continuous unitary representation of a Lie group G in a Hilbert space H and D_∞ the space of C^∞ vectors for U* [3]; *assume that D is a group-invariant dense subspace of H contained in D_∞. Suppose that T is a symmetric operator defined on D_∞ such that $TU(g) \supseteq U(g)T$ for all $g \in G$. Then T is essentially selfadjoint on D.*

PROOF. From [3, Theorem 1.3] it follows immediately that D is dense in D_∞ with its natural Fréchet topology, and from [3, Corollary 2.2] that T is

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essentially selfadjoint. Let (D_∞, τ) denote D_∞ equipped with its natural Fréchet topology and consider T as an operator S from (D_∞, τ) to H; since T is symmetric it is closable and it follows easily that S is closed. By the closed graph theorem S is actually continuous, so for each $y \in D_\infty$ there is a sequence $\{x_n\}_{n \in N} \subseteq D$ such that $\{x_n\}$ converges to y in D_∞ and $\{Tx_n\}$ converges to Ty in H. But this means that $T \upharpoonright D_\infty \subseteq (T \upharpoonright D)^-$, and, hence, $(T \upharpoonright D)^- = (T \upharpoonright D_\infty)^-$ is selfadjoint. Q.E.D.

COROLLARY 2. *Let T be a symmetric operator in $H = L^2(\xi)$ defined on the C^∞ vectors for the group $V(t)$ above. Suppose that $TV(t) \supseteq V(t)T$ ($T \in \mathbf{R}$). Then T is essentially selfadjoint on $C_0^\infty(\xi)$.*

PROOF. As the domain D we take $C_0^\infty(\xi)$ and directly apply the lemma. Q.E.D.

If we interpret iL as the hamiltonian of a quantum mechanical system with Hilbert state space H, we see that the result asserts the essential selfadjointness of those constants of motion which are defined on the C^∞ vectors for $V(t)$.

The lemma above may also be applied to the case of a second order wave equation to yield essential selfadjointness of operators commuting with $-\Delta + g(x)$ on $\mathbf{R}^n$. Specifically we have

COROLLARY 3. *Let g be C^∞ and $T = -\Delta + g(x)$ semibounded on $L^2(\mathbf{R}^n)$, and let D_∞ be the largest subspace where all powers of T are defined. Suppose T_1 is symmetric and defined on D_∞ such that $T_1 T = TT_1$ on D_∞. Then T_1 is essentially selfadjoint on $C_0^\infty(\mathbf{R}^n)$.*

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