A NOTE ON THE ESSENTIAL SELFADJOINTNESS OF CLASSICAL CONSTANTS OF MOTION

BENT ØRSTED

Abstract. It is shown how the results by Chernoff [1] and also Rauch and Taylor [2] on the essential selfadjointness of powers of generators of hyperbolic mixed problems can be combined with results by Poulsen [3] to give essential selfadjointness of symmetric operators commuting with the hyperbolic problem, as specified below.

Let $M$ be a complete Riemannian manifold with volume element $dv$ and $\xi$ a hermitian bundle over $M$ with $\langle \cdot, \cdot \rangle_x$ the inner product in the fiber over $x$. As in [1] we consider a first-order symmetric hyperbolic operator $L$ on $C^\infty(\xi)$, all $C^\infty$ sections of $\xi$, of the form

$$L = \sum_{i=1}^{n} A_i(x) \frac{\partial}{\partial x_i} + B(x).$$

Under the assumption that the local velocity of propagation of solutions to $\frac{\partial u}{\partial t} = Lu$ is sufficiently low, we get [1] that this has unique global solutions given initial data in $C^\infty_0(\xi)$, all $C^\infty$ sections with compact support. Hence we infer the existence of a continuous unitary one-parameter group $V(t)$ in $H = L^2(\xi)$ such that

1. $\frac{d}{dt}V(t)u = VL(t)u = V(t)Lu,$
2. $V(t)u \in C^\infty_0(\xi)$

for all $u \in C^\infty_0(\xi)$. In particular, the generator of $V(t)$ contains $iL$.

Lemma 1. Let $U$ be a continuous unitary representation of a Lie group $G$ in a Hilbert space $H$ and $D_\infty$ the space of $C^\infty$ vectors for $U$ [3]; assume that $D$ is a group-invariant dense subspace of $H$ contained in $D_\infty$. Suppose that $T$ is a symmetric operator defined on $D_\infty$ such that $TU(g) \supseteq U(g)T$ for all $g \in G$. Then $T$ is essentially selfadjoint on $D$.

Proof. From [3, Theorem 1.3] it follows immediately that $D$ is dense in $D_\infty$ with its natural Fréchet topology, and from [3, Corollary 2.2] that $T$ is...
essentially selfadjoint. Let \((D_\infty, \tau)\) denote \(D_\infty\) equipped with its natural Fréchet topology and consider \(T\) as an operator \(S\) from \((D_\infty, \tau)\) to \(H\); since \(T\) is symmetric it is closable and it follows easily that \(S\) is closed. By the closed graph theorem \(S\) is actually continuous, so for each \(y \in D_\infty\) there is a sequence \(\{x_n\}_{n \in \mathbb{N}} \subseteq D\) such that \(\{x_n\}\) converges to \(y\) in \(D_\infty\) and \(\{Tx_n\}\) converges to \(Ty\) in \(H\). But this means that \(T \uparrow D_\infty \subseteq (T \uparrow D)^\perp\), and, hence, \((T \uparrow D)^\perp = (T \uparrow D_\infty)^\perp\) is selfadjoint. Q.E.D.

**Corollary 2.** Let \(T\) be a symmetric operator in \(H = L^2(\xi)\) defined on the \(C^\infty\) vectors for the group \(V(t)\) above. Suppose that \(TV(t) \supseteq V(t)T\) \((T \in \mathbb{R})\). Then \(T\) is essentially selfadjoint on \(C^\infty_0(\xi)\).

**Proof.** As the domain \(D\) we take \(C^\infty_0(\xi)\) and directly apply the lemma. Q.E.D.

If we interpret \(iL\) as the Hamiltonian of a quantum mechanical system with Hilbert state space \(H\), we see that the result asserts the essential selfadjointness of those constants of motion which are defined on the \(C^\infty\) vectors for \(V(t)\).

The lemma above may also be applied to the case of a second order wave equation to yield essential selfadjointness of operators commuting with \(-\Delta + g(x)\) on \(\mathbb{R}^n\). Specifically we have

**Corollary 3.** Let \(g\) be \(C^\infty\) and \(T = -\Delta + g(x)\) semibounded on \(L^2(\mathbb{R}^n)\), and let \(D_\infty\) be the largest subspace where all powers of \(T\) are defined. Suppose \(T_1\) is symmetric and defined on \(D_\infty\) such that \(T_1 T = TT_1\) on \(D_\infty\). Then \(T_1\) is essentially selfadjoint on \(C^\infty_0(\mathbb{R}^n)\).

**References**

