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ON WEAK SEQUENTIAL COMPLETENESS IN BIPROJECTIVE TENSOR PRODUCT SPACES

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ABSTRACT. In this article we are dealing with a study of the weak sequential completeness of $E \hat{\otimes}_\varepsilon F$, the completed ε-tensor product of suitable locally convex spaces E and F. In particular, certain results of Lewis [3] are extended.

1. **Introduction.** Let E and F be complete locally convex spaces, $E \hat{\otimes}_\varepsilon F$ the respective completed (biprojective) ε-tensor product space and let T be the (vector) subspace of $(E' \otimes F')^*$, consisting of the $\sigma((E' \otimes F')^*, E' \otimes F')$-limits of all weakly Cauchy sequences in $E \hat{\otimes}_\varepsilon F$. In this note it is proved that $E \hat{\otimes}_\varepsilon F$ is weakly sequentially complete if, and only if, E and F are weakly sequentially complete and, moreover, every linear map $u \in T$ transfers equicontinuous subsets of E' into relatively compact subsets of F (cf. Theorem 3.2 below). From this result a number of corollaries, referred to the weak sequential completeness of the spaces $E \hat{\otimes}_\varepsilon F$, $\mathcal{L}_c(E, F)$, $\mathcal{L}_b(E, F)$ and $\mathcal{L}_e(E'_\tau, F)$, are derived. They have a special bearing, into the case under consideration, on certain results of [3, §2], formulated for Banach spaces (with the metric approximation property), which also motivated the present work.

2. **Notations and preliminaries.** All vector spaces considered in the following are over the field $\mathbf{C}$ of complex numbers. The topological spaces involved are assumed to be Hausdorff. For a dual pair $\langle X, Y\rangle$ of vector spaces we denote by $\sigma(X, Y)$ and $\tau(X, Y)$ the weak and the Mackey topology of X respectively. The locally convex spaces thus obtained are denoted by X_σ and X_τ. If E is a locally convex space, we denote by E^* and E' the algebraic and the topological dual of E respectively. Moreover, we consider the topology $c(E', E)$ of the absolutely convex compact convergence in E and denote by E'_c the respectively locally convex space. For two locally convex spaces E and F, we denote by $L(E, F)$ and $\mathcal{L}(E, F)$ the (vector) space of linear maps and continuous linear maps from E into F respectively. $\mathcal{L}_c(E, F)$ (resp. $\mathcal{L}_b(E, F)$) is the space $\mathcal{L}(E, F)$, equipped with the topology of absolutely convex compact (resp. bounded) convergence in E. On the other hand, we consider the (vector) space $K(E', F)$ of all linear maps from E' into F, which transfers

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equicontinuous subsets of E' into relatively compact subsets of F and denote by $\mathcal{K}(E', F)$ its (vector) subspace of all weakly ($\sigma(E', E)$, $\sigma(F, F')$) continuous linear maps. Now we say that a Banach space E has the *metric* (*s*)-*approximation property* if, for all $\varepsilon > 0$ and $y_1, \ldots, y_m \in E$, there is a finite rank operator u on E with $\|u\| \leqslant 1$ and $\|u(y_i) - y_i\| \leqslant \varepsilon$ for all $i = 1, \ldots, m$ [6, p. 9].

On the other hand, we need for the sake of references in the sequel the following result of [8] which extends it [3, Lemmas 2.1 and 2.2].

LEMMA 2.1. *Let $\langle X, Y\rangle$ be a dual pair of vector spaces and let $\mathfrak{S}$ be a saturated cover of Y, consisting of $\sigma(Y, X)$-bounded and $\sigma(X^*, X)$-closed subsets of Y. We also consider the locally convex topology e of $\mathfrak{S}$-convergence on X, X' the topological dual of (X, e), the completion $\hat{X} := (\hat{X}, \hat{e})$ of (X, e) and a (topological vector) subspace H of $\hat{X}$ with $X \subseteq H \subseteq \hat{X}$. Then the topologies $\sigma(H, Y)$ and $\sigma(H, X')$ have the same $\hat{e}$-bounded null sequences (and hence also the same $\hat{e}$-bounded Cauchy sequences) in H.*

3. **Weak sequential completeness in biprojective tensor products.** Let E and F be locally convex spaces and let $\mathcal{L}(E'_\tau, F)$ be the (vector) space of continuous linear maps of E'_τ into F, which, of course, coincides with the (vector) space $\mathcal{L}(E'_\sigma, F_\sigma)$ of continuous linear maps of E'_σ into F_σ (cf. [7, p. 429, Proposition 42.2]). Furthermore, we denote by $\mathcal{L}_e(E'_\tau, F)$ the space $\mathcal{L}(E'_\tau, F)$, equipped with the topology of equicontinuous convergence in E' and consider the (topological vector) subspace $\mathcal{L}_e(E'_c, F)$ of it, where $\mathcal{L}(E'_c, F)$ is the (vector) space of continuous linear maps from E'_c into F. By [5, p. 35, Proposition 5], $\mathcal{L}(E'_c, F) = \mathcal{K}(E', F)$ (§2). For a map $u \in L(E, F)$ we denote by ${}^t u$ its transpose. If $u \in \mathcal{L}(E'_\tau, F)$, then ${}^t u \in \mathcal{L}(F'_\tau, E)$. Now let $E \otimes F$ be the tensor product (vector) space of E and F. Then the respective biprojective (locally convex) topology is the topology ε of $\mathfrak{S}$-convergence on $E \otimes F$, where $\mathfrak{S}$ is the family of the sets $A' \otimes B'$ with A' and B' weakly closed equicontinuous subsets of E' and F' respectively. The locally convex space $E \otimes_\varepsilon F$ thus obtained is referred to as the (respective) *biprojective tensor product space*. We denote by $E \hat{\otimes}_\varepsilon F$ the completion of $E \otimes_\varepsilon F$ and by $\hat{\varepsilon}$ the topology of $E \hat{\otimes}_\varepsilon F$. $E \hat{\otimes}_\varepsilon F$ is contained in $\mathcal{L}_e(E'_c, F)$ whenever the last space is complete. Examples of locally convex spaces E and F with $\mathcal{L}_e(E'_c, F)$ complete are treated in [1] and [2].

Now for any element u of $\mathcal{L}(E'_\tau, F)$ and for every $x' \in E'$ and $y' \in F'$, one has

$$\langle u, x' \otimes y'\rangle = \langle u(x'), y'\rangle = \langle x', {}^t u(y')\rangle. \tag{3.1}$$

On the other hand, if (u_n) is a (weakly) $\sigma(E \hat{\otimes}_\varepsilon F, (E \hat{\otimes}_\varepsilon F)')$-Cauchy sequence in $E \hat{\otimes}_\varepsilon F$, then by Grothendieck's completeness theorem, (u_n) is contained in $(E' \otimes F')^*$ and it is also clearly $\sigma((E' \otimes F')^*, E' \otimes F')$-Cauchy, so that, by the completeness of

$$\left((E' \otimes F')^*, \sigma\left((E' \otimes F')^*, E' \otimes F'\right)\right),$$

there exists the $\sigma((E' \otimes F')^*, E' \otimes F')$-limit of (u_n). Thus, let T denote the (vector) subspace of

$$(E' \otimes F')^* = L(E', F'^*)$$

consisting of the $\sigma((E' \otimes F')^*, E' \otimes F')$-limits of all (weakly) $\sigma(E \hat{\otimes}_\varepsilon F, (E \hat{\otimes}_\varepsilon F)')$-Cauchy sequences in $E \hat{\otimes}_\varepsilon F$.

We now need

LEMMA 3.1. *If E and F are weakly sequentially complete locally convex spaces with $E \hat{\otimes}_\varepsilon F \subseteq \mathfrak{L}(E'_\tau, F)$, then $T \subseteq \mathfrak{L}(E'_\tau, F)$.*

PROOF. Let $u \in T$ and let (u_n) be a $\sigma(E \hat{\otimes}_\varepsilon F, (E \hat{\otimes}_\varepsilon F)')$-Cauchy sequence in $E \hat{\otimes}_\varepsilon F$ with $u_n \to u$ in the (weak) topology

$$\sigma((E' \otimes F')^*, E' \otimes F').$$

By hypothesis of weak sequential completeness there exist clearly linear maps $v \in L(E', F)$ and $w \in L(F', E)$ such that $v(x') = \lim u_n(x')$ in F_σ and $w(y') = \lim {}^t u_n(y')$ in E_σ for every $x' \in E'$ and $y' \in F'$. On the other hand, by hypothesis that $E \hat{\otimes}_\varepsilon F \subseteq \mathfrak{L}(E'_\tau, F)$ and relation (3.1) above, one clearly has ${}^t v = w$, that is ${}^t v(F') \subseteq E$ and, hence, v belongs to $\mathfrak{L}(E'_\sigma, F_\sigma) = \mathfrak{L}(E'_\tau, F)$. But it obviously follows that $u_n \to v$ in the (weak) topology

$$\sigma((E' \otimes F')^*, E' \otimes F')$$

so that $u = v$, which proves the assertion.

We are now in a position to state and prove the main result of this note. That is, we have

THEOREM 3.2. *Let E and F be complete locally convex spaces and let T be the (vector) subspace of*

$$(E' \otimes F')^* = L(E', F'^*)$$

consisting of the $\sigma((E' \otimes F')^*, E' \otimes F')$-limits of all (weakly) $\sigma(E \hat{\otimes}_\varepsilon F, (E \hat{\otimes}_\varepsilon F)')$-Cauchy sequences in $E \hat{\otimes}_\varepsilon F$. On the other hand, we consider the (vector) space $K(E', F)$ of all linear maps from E' into F which transfers equicontinuous subsets of E' into relatively compact subsets of F, and the following statements:

(1) (a) *E and F are weakly sequentially complete.*

(b) *T is contained in $K(E', F)$.*

(2) *$E \hat{\otimes}_\varepsilon F$ is weakly sequentially complete.*

Then (2) *implies* (1). *If, moreover, one of the spaces E, F and E'_c has the approximation property, then* (1) *implies also* (2).

PROOF. (1) implies (2). Let (u_n) be a $\sigma(E \hat{\otimes}_\varepsilon F, (E \hat{\otimes}_\varepsilon F)')$-Cauchy sequence in $E \hat{\otimes}_\varepsilon F$ and $u \in T$ with $u_n \to u$ in the (weak) topology

$$\sigma((E' \otimes F')^*, E' \otimes F').$$

By hypothesis (1)(a) and Lemma 3.1 above, u belongs to $\mathfrak{L}(E'_\sigma, F_\sigma)$ so that by hypothesis (1)(b) and [5, p. 35, Proposition 5] u belongs also to $\mathfrak{L}(E'_c, F)$. On the other hand, by hypotheses and [1, p. 197, Satz 9], u belongs to $E \hat{\otimes}_\varepsilon F$ and $u_n \to u$ in the (weak) topology $\sigma(E \hat{\otimes}_\varepsilon F, (E \hat{\otimes}_\varepsilon F)')$ (Lemma 2.1), and hence $E \hat{\otimes}_\varepsilon F$ is weakly sequentially complete. Now (2) implies (1). In fact by standard arguments, given, for instance, in the proof of [4, p. 167, §9.1], it follows that (2) implies (1)(a). On the other hand, by the fact that $T = E \hat{\otimes}_\varepsilon F$ and the techniques of (1) implies (2) above, (2) clearly implies (1)(b), and the proof is completed.

In particular, the following corollaries have a special bearing on respective results of [3, §2].

COROLLARY 3.3. *Let E and F be complete locally convex spaces such that $\mathcal{L}(E'_\sigma, F_\sigma) = \mathcal{K}(E', F)$. Suppose, moreover, that one of the spaces E, F and E'_c has the approximation property. Then $E \hat{\otimes}_\varepsilon F$ $(= \mathcal{L}_e(E'_c, F) = \mathcal{L}_e(E'_\tau, F))$ is weakly sequentially complete if (and only if) both E and F are.*

PROOF. It follows by [1, p. 197, Satz 9] and Theorem 3.2 above.

By Theorem 3.2 and the corresponding techniques of [3, p. 203, Theorem 2.1], we clearly get

COROLLARY 3.4. *Let E and F be Banach spaces such that E or F have the metric (s)-approximation property and let $E \hat{\otimes}_\varepsilon F$ be weakly sequentially complete. Then $\mathcal{L}(E'_\tau, F) = \mathcal{K}(E', F)$.*

On the other hand, by Theorem 3.2 in the foregoing and [2, p. 7, Satz 6] we also get

COROLLARY 3.5. *Let E and F be locally convex spaces such that E is quasi-complete bornological (resp. Montel (LF)-space) with the approximation property and F is complete. Then $\mathcal{L}_c(E, F) = E'_c \hat{\otimes}_\varepsilon F$ (resp. $\mathcal{L}_b(E, F) = E'_b \hat{\otimes}_\varepsilon F$) is weakly sequentially complete if (and only if) both E'_c and F (resp. E'_b and F) are.*

Now if E and F are complete locally convex spaces with F a semi-Montel space which has the approximation property, then by standard arguments, given, for example, in [7, p. 522, Proposition 50.4], it follows that $E \hat{\otimes}_\varepsilon F = \mathcal{L}_e(E'_\tau, F)$, so that by the foregoing we finally have

COROLLARY 3.6. *Let E and F be complete locally convex spaces such that F is a semi-Montel space which has the approximation property (or, in particular, F is a nuclear space). Then $E \hat{\otimes}_\varepsilon F = \mathcal{L}_e(E'_\tau, F)$ is weakly sequentially complete if (and only if) this is the case for E.*

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