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A REMARK ON THE STRONG LAW OF LARGE NUMBERS

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ABSTRACT. Let $X_1, X_2, \ldots$ be mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \ldots$. For each $n = 1, 2, \ldots$, let $S_n = \sum_{j=1}^n X_j$; then, by the Kolmogorov criterion for mutually independent random variables, $S_n/n^{1/2+\alpha} \to 0$ almost surely as $n \to \infty$ for any positive constant α. A deeper understanding of this theorem will be facilitated if we know the order of magnitude of $E\{N_\infty(\alpha, \varepsilon)\}$ as $\varepsilon \to 0^+$, where $N_\infty(\alpha, \varepsilon)$ is the integer-valued random variable defined by $N_\infty(\alpha, \varepsilon) = \sum_{n=1}^\infty \chi_{(\varepsilon n^{1/2+\alpha}, \infty)}(|S_n|)$. The present note does the work for a wide class of random variables by using Esseen's theorem and Katz-Petrov's theorem.

Let $X_1, X_2, \ldots$ be mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \ldots$. For each $n = 1, 2, \ldots$, let $S_n = \sum_{j=1}^n X_j$; then, by the Kolmogorov criterion for mutually independent random variables, $S_n/n^{1/2+\alpha} \to 0$ almost surely as $n \to \infty$ for any positive constant α. For any positive constants α and ε, let $A_n(\alpha, \varepsilon) = \chi_{(\varepsilon n^{1/2+\alpha}, \infty)}(|S_n|)$ for all $n = 1, 2, \ldots$ and let $N_\infty(\alpha, \varepsilon) = \sum_{n=1}^\infty A_n(\alpha, \varepsilon)$. Then, for a deeper understanding of the theorem above, we will study the order of magnitude of $E\{N_\infty(\alpha, \varepsilon)\}$ as $\varepsilon \to 0^+$, and this is just the main purpose of this note. We start with the following useful lemmas.

LEMMA 1. *Suppose that $X_1, X_2 \ldots$ are mutually independent, standard normal random variables, and α, ε are two positive constants. Then, we have*

$$C_\alpha \varepsilon^{-1/\alpha} - 1 \leqslant E\{N_\infty(\alpha, \varepsilon)\} \leqslant C_\alpha \varepsilon^{-1/\alpha}, \tag{1}$$

where $C_\alpha = \pi^{-1/2} 2^{1/2\alpha} \Gamma(1/2 + 1/2\alpha)$.

PROOF. For any positive integer m, let $N_m(\alpha, \varepsilon) = \sum_{n=1}^m A_n(\alpha, \varepsilon)$; then $1 + E\{N_m(\alpha, \varepsilon)\} = 2 \sum_{n=0}^m \Phi(-\varepsilon n^\alpha)$, where $\Phi(x)$ is the distribution function of the standard normal random variable. By the Euler-Maclaurin sum formula (see [1, pp. 124–125]),

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$$2\sum_{n=0}^{m}\Phi(-\varepsilon n^{\alpha})$$
$$=\frac{1}{2}+\Phi(-\varepsilon m^{\alpha})+2\int_0^m\Phi(-\varepsilon x^{\alpha})\,dx-2\int_0^m P_1(x)\,d\Phi(-\varepsilon x^{\alpha}),$$

where $P_1(x)=[x]-x+1/2$ and $[x]$ denotes the greatest integer not exceeding x. Letting $m\to\infty$ and by the monotone convergence theorem, we get

$$\frac{1}{2}+E\{N_\infty(\alpha,\varepsilon)\}=2\int_0^\infty\Phi(-\varepsilon x^{\alpha})\,dx-2\int_0^\infty P_1(x)\,d\Phi(-\varepsilon x^{\alpha}),$$

hence

$$2\int_0^\infty\Phi(-\varepsilon x^{\alpha})\,dx+\int_0^\infty d\Phi(-\varepsilon x^{\alpha})$$
$$\leqslant\frac{1}{2}+E\{N_\infty(\alpha,\varepsilon)\}$$
$$\leqslant 2\int_0^\infty\Phi(-\varepsilon x^{\alpha})\,dx-\int_0^\infty d\Phi(-\varepsilon x^{\alpha}).$$

Therefore,

$$c_\alpha\varepsilon^{-1/\alpha}-1\leqslant E\{N_\infty(\alpha,\varepsilon)\}\leqslant C_\alpha\varepsilon^{-1/\alpha}.$$

For Lemmas 2 and 3, let $X_1, X_2,\ldots$ be mutually independent random variables such that $E(X_n)=0$, $E(X_n^2)=\sigma_n^2<\infty$ (not all of σ_n^2's are zero), $S_n=\sum_{j=1}^n X_j$, $B_n^2=\sum_{j=1}^n\sigma_j^2$, $\Phi_n(x)=P\{S_n/B_n\leqslant x\}$ for all $n=1,2,\ldots,$ and let G be the class of nonnegative functions $g(x)$ satisfying the following conditions: (i) $g(x)$ is nondecreasing on the interval $(0,\infty)$, is even on $(-\infty,\infty)$ and $g(x)\to\infty$ as $x\to\infty$; (ii) the function $x/g(x)$ does not decrease on $(0,\infty)$.

LEMMA 2 (PETROV). *If there exists a function g in G such that $E\{X_n^2g(X_n)\}<\infty$ for all $n=1,2,\ldots,$ then there exists an absolute constant C_1^* such that*

$$\Delta(n)=\sup_{-\infty<x<\infty}|\Phi_n(x)-\Phi(x)|\leqslant\frac{C_1^*}{B_n^2g(B_n)}\sum_{j=1}^n E\{X^2g(X_j)\}.\tag{2}$$

PROOF. See [4, pp. 242–244].

LEMMA 3 (ESSEEN). *If $\Delta(n)\leqslant\frac{1}{2}$ for all $n>n_0$, then there exists an absolute constant C_2^* such that*

$$|\Phi_n(x)-\Phi(x)|\leqslant\min\left\{\Delta(n),C_2^*\cdot\frac{\Delta(n)\log(1/\Delta(n))}{1+x^2}\right\},\tag{3}$$

for all $n>n_0$ and all values of x.

PROOF. See [2, pp. 68–70].

Now we are in the position to state and prove our main theorem and corollaries.

THEOREM 1. *Suppose that $X_1, X_2, \ldots$ are mutually independent random variables such that $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 = 1$ for all $n = 1, 2, \ldots$. If there exists a function g in G such that*

(i) $\limsup_{n\to\infty} n^{-1} \sum_{j=1}^{n} E\{X_j^2 g(X_j)\} < \infty$,

(ii) $\sum_{n=1}^{\infty} (\log n)/n^{2\alpha} g(\sqrt{n}) < \infty$ *for some constant* α $(0 < \alpha \leqslant \frac{1}{2})$, *then, we have*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1/\alpha} E\{N_\infty(\alpha, \varepsilon)\} = C_\alpha. \tag{4}$$

PROOF. The proof will be given by the following two steps.

Step 1. Assume that $X_1, X_2, \ldots$ are i.i.d. normal random variables with mean 0 and variance 1. Then, by Lemma 1,

$$C_\alpha \varepsilon^{-1/\alpha} - 1 \leqslant E\{N_\infty(\alpha, \varepsilon)\} \leqslant C_\alpha \varepsilon^{-1/\alpha} \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \varepsilon^{1/\alpha} E\{N_\infty(\alpha, \varepsilon)\} = C_\alpha.$$

Step 2. Assume that $X_1, X_2, \ldots$ and g satisfy the conditions in Theorem 1.

$$\begin{aligned} E\{N_\infty(\alpha, \varepsilon)\} &= \sum_{n=1}^{\infty} P\{|S_n| > \varepsilon n^{1/2+\alpha}\} \\ &= \sum_{n=1}^{\infty} \{P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_\Phi(|S_n| > \varepsilon n^{1/2+\alpha})\} \\ &\quad + \sum_{n=1}^{\infty} P_\Phi(|S_n| > \varepsilon n^{1/2+\alpha}), \end{aligned}$$

where P_Φ denotes the measure induced by the standard normal random variables. By Step 1, it is sufficient to show that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1/\alpha} \sum_{n=1}^{\infty} \{P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_\Phi(|S_n| > \varepsilon n^{1/2+\alpha})\} = 0. \tag{5}$$

By Lemma 2,

$$\Delta(n) = \sup_{-\infty<x<\infty} |\Phi_n(x) - \Phi(x)| \leqslant \frac{C_1^*}{ng(\sqrt{n})} \sum_{j=1}^{n} E\{X_j^2 g(X_j)\}$$

for all $n = 1, 2, \ldots$. where $\Phi_n(x)$ is the distribution function of $S_n/\sqrt{n}$. By the assumptions that $\sum_{j=1}^{n} E\{X_j^2 g(X_j)\} \leqslant O(n)$ and $g(x) \to \infty$ as $x \to \infty$, there exists a positive integer n_0 such that if $n > n_0$, $\Delta(n) \leqslant \frac{1}{2}$. It is obvious that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1/\alpha} \sum_{j=1}^{n_0} \{P(|S_j| > \varepsilon j^{1/2+\alpha}) - P_\Phi(|S_j| > \varepsilon j^{1/2+\alpha})\} = 0.$$

Hence, it is sufficient to show that

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1/\alpha} \sum_{n>n_0} \{P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_\Phi(|S_n| > \varepsilon n^{1/2+\alpha})\} = 0. \tag{6}$$

But

$$|P(|S_n| > \varepsilon n^{1/2+\alpha}) - P_\Phi(|S_n| > \varepsilon n^{1/2+\alpha})| \leqslant |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| + |\Phi_n(\varepsilon n^\alpha) - \Phi(\varepsilon n^\alpha)|$$

for all $n = 1, 2, \ldots$. So it is sufficient to show that

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n > n_0} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| = 0. \tag{7}$$

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n > n_0} |\Phi_n(\varepsilon n^\alpha) - \Phi(\varepsilon n^\alpha)| = 0. \tag{8}$$

Let $A = \{n | n > n_0, \Delta(n) \leqslant n^{-2}\}$ and $A' = \{n | n > n_0, \Delta(n) > n^{-2}\}$. Now if $n \in A$, then $|\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \leqslant \Delta(n) \leqslant n^{-2}$. Hence

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \leqslant \lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A} n^{-2} = 0.$$

Next, if $n \in A'$, then $1/\Delta(n) \leqslant n^2$, $\log(1/\Delta(n)) \leqslant 2 \log n$, and

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| \right\} \\ &\qquad \leqslant \lim_{\varepsilon \to 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{C_2^* \Delta(n) 2 \log n}{1 + \varepsilon^2 n^{2\alpha}} \right\} \\ &\qquad \leqslant \lim_{\varepsilon \to 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{C_1^* C_2^* 2 \log n \sum_{j=1}^n E\{X_j^2 g(X_j)\}}{n g(\sqrt{n})\{1 + \varepsilon^2 n^{2\alpha}\}} \right\} \end{aligned}$$

(the first inequality is implied by Lemma 3 and the second one is implied by Lemma 2).

Since

$$\limsup_{n \to \infty} n^{-1} \sum_{j=1}^n E\{X_j^2 g(X_j)\} < \infty, \qquad 0 < \alpha \leqslant \tfrac{1}{2},$$

and $\sum_{n=1}^\infty (\log n)/n^{2\alpha} g(\sqrt{n}) < \infty$, it is easy to see that

$$\lim_{\varepsilon \to 0^+} \left\{ \varepsilon^{1/\alpha} \sum_{n \in A'} \frac{(2 \log n) \sum_{j=1}^n E\{X_j^2 g(X_j)\}}{n g(\sqrt{n})\{1 + \varepsilon^2 n^{2\alpha}\}} \right\} = 0.$$

Hence,

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} \sum_{n \in A'} |\Phi_n(-\varepsilon n^\alpha) - \Phi(-\varepsilon n^\alpha)| = 0.$$

Therefore, we get (7). Similarly, we can prove that (8) holds and the proof of Theorem 1, now, is complete.

COROLLARY 1. *Suppose that $X_1, X_2 \ldots$ are mutually independent random variables such that $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2 = 1$, and $E(|X_n|^{2+\delta}) \leqslant M < \infty$ for some positive constants δ and M, and for all $n = 1, 2, \ldots$. Then we have*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/\alpha} E\{N_\infty(\alpha, \varepsilon)\} = C_\alpha, \tag{9}$$

where $\frac{1}{2} - \delta/4 < \alpha \leqslant \frac{1}{2}$ *if* $0 < \delta < 1$, $\frac{1}{4} < \alpha \leqslant \frac{1}{2}$ *if* $\delta \geqslant 1$.

A sharper result is

COROLLARY 2. *Suppose that* $X_1, X_2, \ldots$ *are mutually independent random variables such that* $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2 = 1$, *and* $E\{X_n^2(\log^+|X_n|)^{2+\delta}\} \leqslant M < \infty$ *for some positive constants* δ *and* M, *and for all* $n = 1, 2, \ldots$. *Then we have*

$$\lim_{\varepsilon \to 0} \varepsilon^2 E\{N_\infty(\tfrac{1}{2}, \varepsilon)\} = C_{1/2} = 1. \tag{10}$$

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