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ON THE EXTENSION OF CYLINDER MEASURES TO τ-SMOOTH MEASURES IN LINEAR SPACES

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ABSTRACT. We give a necessary and sufficient condition for a cylindrical probability measure in the weak*-dual of an arbitrary l.c.s. to extend to a τ-smooth Borel-measure; this is to a certain extent a "τ-smooth analogue" of the well-known Prohorov extension theorem (cf. [8, Lemma 3]). Finally, we give two examples marking off our result from related ones treated in the literature.

1. **Introduction and notations.** Let E be a locally convex topological vector space (l.c.s.) and E' (resp. E^*) its topological (resp. algebraic) dual. A linear stochastic process (l.s.p.) $(X_t)_{t\in E}$ on E is an indexed collection of random variables X_t on a common probability space $(\Omega, \mathfrak{A}, P)$ such that every path $X(\omega)$ is in E^* (cf. [5]). Corresponding to every l.s.p. $(X_t)_{t\in E}$, we define a finitely-additive probability measure $\mu_{0,X}$ on $\mathfrak{Z}_{E',E}$, the algebra of cylinder sets in E', by $\mu_{0,X}(A) := (P \circ X^{-1})(A_0)\ \forall A \in \mathfrak{Z}_{E',E}$, where A_0 is a cylinder set in E^* such that $A = A_0 \cap E'$. $\mu_{0,X}$ is called the cylindrical measure corresponding to $(X_t)_{t\in E}$.

Let us call a l.s.p. $(X_t)_{t\in E}$ realizable (resp. strongly realizable, resp. very strongly realizable) in E' if $\mu_{0,X}$ is extendable to a σ-additive measure μ_X on $\mathfrak{A}(\mathfrak{Z}_{E',E})$, the σ-field generated by $\mathfrak{Z}_{E',E}$ (resp. to a τ-smooth measure μ_X on $\mathfrak{B}(E'_\sigma)$–the Borel-sets in the weak*-dual E'_σ, resp. to a tight measure μ_X on $\mathfrak{B}(E'_\sigma)$). Realizability has been studied in [3], [6], [7], e.g.

Many workers have treated very-strong-realizability under various assumptions on the underlying l.c.s. E (cf. [1], [2], [4]); for an arbitrary l.c.s. E, certain natural conditions on the finite-dimensional distributions of a l.s.p. characterizing its very-strong-realizability have been given by Yu. V. Prohorov (cf. [8, Lemma 3]).

In this paper we are concerned with strong realizability. In §2 some natural conditions on a l.s.p. characterizing its strong realizability are proved. In §3

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we construct two examples showing that realizability, strong realizability and very strong realizability are actually different from each other.

We now list some notations that will be used in the sequel. Let $\mathfrak{S}_E := \{(t_1, \ldots, t_n) \colon n \in \mathbf{N}, t_i \in E \text{ pairwise distinct}\}$; for $S = (t_1, \ldots, t_n) \in \mathfrak{S}_E$, $\pi_S \colon E' \to \mathbf{R}^{|S|}, f \to (\langle f, t_1 \rangle, \ldots, \langle f, t_n \rangle)$, is a projection of E' into $\mathbf{R}^{|S|}$; Q_{X_S} is the distribution of $(X_{t_1}, \ldots, X_{t_n})$, $\{Q_{X_S} \colon S \in \mathfrak{S}_E\}$ the set of finite-dimensional distributions of $(X_t)_{t \in E}$. By $\mathfrak{K}(E'_\sigma)$, $\mathfrak{F}(E'_\sigma)$, $\mathfrak{B}(E'_\sigma)$, $\mathfrak{Z}(E'_\sigma)$ we denote the compact, closed, Borel-, zero-sets of E'_σ, respectively. A measure will always be understood to be a probability measure. A measure μ on $\mathfrak{B}(E'_\sigma)$ will be called tight (resp. regular) if $\mu(A) = \sup\{\mu(K) \colon A \supset K \in \mathfrak{K}(E'_\sigma)\}$ (resp. $\mu(A) = \sup\{\mu(F) \colon A \supset F \in \mathfrak{F}(E'_\sigma)\}$) $\forall A \in \mathfrak{B}(E'_\sigma)$; it will be called τ-smooth (resp. τ-smooth at $\varnothing$), if $\mu(\bigcap_{i \in I} F_i) = \inf_{i \in I} \mu(F_i)$ for every system $\{F_i \colon i \in I\} \subset \mathfrak{F}(E'_\sigma)$ that is filtering downward (resp. filtering downward to $\varnothing$).

For a measure μ and a system of sets $\mathfrak{M}$, $\mu|\mathfrak{M}$ denotes the restriction of μ to $\mathfrak{M}$ (if this makes sense); $\mathfrak{A}(\mathfrak{M})$ is the smallest σ-algebra containing $\mathfrak{M}$; μ_* (resp. μ^*) is the inner (resp. outer) measure pertaining to μ. χ_A is the characteristic function of the set A. For $A \subset \mathbf{R}^n$ (resp. $A \subset E'_\sigma$), $\overline{A}$ (resp. $\overline{A}^\sigma$) denotes the closure of A in $\mathbf{R}^n$ (resp. E'_σ). lin (M) is the linear hull of M (over the reals).

2. **Existence of τ-smooth measures on $\mathfrak{B}(E'_\sigma)$.** The following theorem is the main result of this paper. It is proved by an application of an extension theorem due to F. Topsøe.

THEOREM. *Let E be a l.c.s. and $(X_t)_{t \in E}$ a l.s.p.; $(X_t)_{t \in E}$ is strongly realizable in E' iff the following two conditions are fulfilled*:

(i) *for every increasing sequence $(F_n)_{n \in \mathbf{N}}$ in $\mathfrak{Z}(E'_\sigma)$ with $\bigcup_{n \in \mathbf{N}} F_n \in \mathfrak{Z}(E'_\sigma)$ we have*

$$\sup_{n \in \mathbf{N}} \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_n)}) = \inf_{S \in \mathfrak{S}_E} Q_{X_S}\left(\overline{\pi_S\left(\bigcup_{n \in \mathbf{N}} F_n\right)}\right);$$

(ii) *for every system $\{F_j \colon j \in J\} \subset \mathfrak{Z}(E'_\sigma)$ filtering downward to $\varnothing$ we have*

$$\inf_{j \in J} \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_j)}) = 0.$$

PROOF. We shall use the following easily verified result:

$$(*) \qquad F = \bigcap_{S \in \mathfrak{S}_E} \pi_S^{-1}(\overline{\pi_S(F)}) \quad \forall F \in \mathfrak{F}(E'_\sigma).$$

"$\Rightarrow$": $\mu_X|\mathfrak{B}(E'_\sigma)$ τ-smooth, $(*) \Rightarrow$

$$(+) \qquad \begin{aligned} \mu(F) &= \mu\left(\bigcap_{S \in \mathfrak{S}_E} \pi_S^{-1}(\overline{\pi_S(F)})\right) = \inf_{S \in \mathfrak{S}_E} \mu(\pi_S^{-1}(\overline{\pi_S(F)})) \\ &= \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F)}) \quad \forall F \in \mathfrak{Z}(E'_\sigma) \subset \mathfrak{F}(E'_\sigma). \end{aligned}$$

By (+), we obtain (i) and (ii) from the assumptions.

"$\Leftarrow$": We define

$$(++) \qquad \mu(F) := \inf_{S\in\mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F)}) \quad \forall F \in \mathfrak{T}(E'_\sigma).$$

Since E'_σ is completely regular, the set of arbitrary intersections of elements of $\mathfrak{T}(E'_\sigma)$ coincides with $\mathfrak{F}(E'_\sigma)$ and every τ-smooth measure on $\mathfrak{B}(E'_\sigma)$ is regular. Therefore, as a special case of [11, Theorem 4.2], we obtain: if $\mu|\mathfrak{T}(E'_\sigma)$ is a tight content and τ-smooth at $\varnothing$, then μ has an extension to a τ-smooth measure on $\mathfrak{B}(E'_\sigma)$. By (++) and (ii), $\mu|\mathfrak{T}(E'_\sigma)$ is τ-smooth at $\varnothing$. Thus we have to show that it is a tight content, i.e.

$$(**) \qquad \mu(F_1) + \sup\{\mu(F)\colon F \subset F_2\backslash F_1, F \in \mathfrak{T}(E'_\sigma)\} = \mu(F_2), \quad \text{if } F_1, F_2 \in \mathfrak{T}(E'_\sigma) \text{ and } F_1 \subset F_2.$$

To this end, we shall need

$$(***) \quad F_1, F_2 \in \mathfrak{T}(E'_\sigma),\ F_1 \cap F_2 = \varnothing \Rightarrow \mu(F_1) + \mu(F_2) = \mu(F_1 \cup F_2).$$

ad (***):
$$F_1 \cap F_2 = \varnothing \underset{(*)}{\Rightarrow} \bigcap_{S\in\mathfrak{S}_E} \pi_S^{-1}(\overline{\pi_S(F_1)}) \cap \bigcap_{S\in\mathfrak{S}_E} \pi_S^{-1}(\overline{\pi_S(F_2)}) = \varnothing$$
$$\Leftrightarrow \bigcap_{S\in\mathfrak{S}_E} \pi_S^{-1}(\overline{\pi_S(F_1)} \cap \overline{\pi_S(F_2)}) = \varnothing.$$

Thus $\{\pi_S^{-1}(\overline{\pi_S(F_1)} \cap \overline{\pi_S(F_2)})\colon S \in \mathfrak{S}_E\}$ is filtering downward to $\varnothing$; since it is contained in $\mathfrak{T}(E'_\sigma)$, (ii) yields

$$\begin{aligned} 0 &= \inf_{S_1\in\mathfrak{S}_E} \inf_{S_2\in\mathfrak{S}_E} Q_{X_{S_2}}(\overline{\pi_{S_2}(\pi_{S_1}^{-1}(\overline{\pi_{S_1}(F_1)} \cap \overline{\pi_{S_1}(F_2)}))}) \\ &= \inf_{S\in\mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(\pi_S^{-1}(\overline{\pi_S(F_1)} \cap \overline{\pi_S(F_2)}))}) \\ &= \inf_{S\in\mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_1)} \cap \overline{\pi_S(F_2)}), \end{aligned}$$

as is easily seen. For $\varepsilon > 0$ we choose $S_0 = S_0(\varepsilon) \in \mathfrak{S}_E$ such that

$$Q_{X_{S_0}}(\overline{\pi_{S_0}(F_1 \cup F_2)}) \leqslant \inf_{S\in\mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_1 \cup F_2)}) + \varepsilon.$$

Now we can find $S_1 = S_1(\varepsilon) \in \mathfrak{S}_E$ such that $S_1 \supset S_0$ and

$$Q_{X_{S_1}}(\overline{\pi_{S_1}(F_1)} \cap \overline{\pi_{S_1}(F_2)}) \leqslant \varepsilon,$$

thus obtaining

$$\begin{aligned}\mu(F_1) + \mu(F_2) &\leqslant Q_{X_{S_1}}(\overline{\pi_{S_1}(F_1)}) + Q_{X_{S_1}}(\overline{\pi_{S_1}(F_2)}) \\ &= Q_{X_{S_1}}(\overline{\pi_{S_1}(F_1)} \cup \overline{\pi_{S_1}(F_2)}) + Q_{X_{S_1}}(\overline{\pi_{S_1}(F_1)} \cap \overline{\pi_{S_1}(F_2)}) \\ &\leqslant Q_{X_{S_0}}(\overline{\pi_{S_0}(F_1)} \cup \overline{\pi_{S_0}(F_2)}) + \varepsilon = Q_{X_{S_0}}(\overline{\pi_{S_0}(F_1 \cup F_2)}) + \varepsilon \\ &\leqslant \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_1 \cup F_2)}) + 2\varepsilon.\end{aligned}$$

This means $\mu(F_1) + \mu(F_2) \leqslant \mu(F_1 \cup F_2)$. On the other hand,

$$\begin{aligned}\mu(F_1 \cup F_2) &= \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_1 \cup F_2)}) \\ &\leqslant \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_1)}) + \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(F_2)}) \\ &= \mu(F_1) + \mu(F_2).\end{aligned}$$

This completes the proof of (***).

Next, we can show (**).

(***) and monotonicity of $\mu|\mathfrak{F}(E'_\sigma)$ imply the validity of "$\leqslant$" in (**). To prove "$\geqslant$", we note that there is an increasing sequence $(\tilde{F}_n)_{n\in\mathbf{N}}$ in $\mathfrak{F}(E'_\sigma)$ such that $\bigcup_{n\in\mathbf{N}}\tilde{F}_n = F_2 \backslash F_1$. We obtain

$$\mu(F_1) + \sup_{n\in\mathbf{N}} \mu(\tilde{F}_n) = \sup_{n\in\mathbf{N}} (\mu(F_1) + \mu(\tilde{F}_n)) \underset{(***)}{=} \sup_{n\in\mathbf{N}} \mu(F_1 \cup \tilde{F}_n) \underset{(\mathrm{i}),(++)}{=} \mu(F_2).$$

Consequently, "$\geqslant$" holds true; this completes the proof of (**), and thereby shows the existence of a τ-smooth extension of μ to $\mathfrak{B}(E'_\sigma)$. This extension will still be called μ.

Now it remains to be shown:

$$(+++) \qquad \mu(A) = \mu_{0,X}(A) \quad \forall A \in \mathfrak{Z}_{E',E}.$$

ad(+++): w.l.o.g. we may suppose $A = \pi_{S_0}^{-1}(\pi_{S_0}(A))$ with $S_0 = (t_1, \ldots, t_m) \in \mathfrak{S}_E$ such that the t_i $(1 \leqslant i \leqslant m)$ are linearly independent and with $\pi_{S_0}(A) \subset \mathbf{R}^m$ closed. Thus $A \in \mathfrak{F}(E'_\sigma)$ and

$$\mu_{0,X}(A) = Q_{X_{S_0}}(\pi_{S_0}(A)) = \inf_{S \in \mathfrak{S}_E} Q_{X_S}(\overline{\pi_S(A)}) = \mu(A),$$

by (++). This implies (+++) by the uniqueness statements of well-known extension theorems. Putting $\mu_X|\mathfrak{B}(E'_\sigma) := \mu|\mathfrak{B}(E'_\sigma)$, the proof of the theorem is complete.

REMARK. There is at most one τ-smooth measure $\mu_X|\mathfrak{B}(E'_\sigma)$ extending $\mu_{0,X}$, as is easily seen (from the "$\Rightarrow$"-direction of the above proof, e.g.).

3. Examples.

(3.1) EXAMPLE. There is a l.c.s. E and l.s.p. $(X_t)_{t\in E}$ s.t. the following hold true:

(a) $(X_t)_{t\in E}$ is strongly realizable in E',

(b) $(X_t)_{t\in E}$ is not very strongly realizable in E'.

PROOF. We consider the probability space $([0,1], \mathfrak{B}, \lambda)$ with $\mathfrak{B}$ denoting the Borel-sets in the unit interval $[0,1]$ and λ Lebesgue-measure. As is well known, there is a set $B \subset [0,1]$ such that $\lambda^*(B)$ (resp. $\lambda_*(B)$) is 1 (resp. 0), and a measure $\tilde{\lambda}$ on $([0,1], \mathfrak{A}(\mathfrak{B} \cup \{B\}))$ extending $\lambda|\mathfrak{B}$ with $\tilde{\lambda}(B) = 1$.

Next we construct E. For every $0 < \varepsilon \in \mathbf{Q}$ and every $q \in \mathbf{Q} \cap [0,1]$, define $t_{q,\varepsilon} \in \mathbf{R}^{[0,1]}$ by

$$t_{q,\varepsilon}(x) := \begin{cases} \varepsilon^{-1}(x - q + \varepsilon) & \forall x \in [q-\varepsilon, q] \cap [0,1], \\ \varepsilon^{-1}(q + \varepsilon - x) & \forall x \in [q, q+\varepsilon] \cap [0,1], \\ 0 & \forall x \notin]q-\varepsilon, q+\varepsilon[\cap [0,1]. \end{cases}$$

Let $(t'_k)_{k\geqslant 2}$ be a denumeration of $T: = \{t_{q,\varepsilon}: q \in \mathbf{Q} \cap [0,1], \mathbf{Q} \ni \varepsilon > 0\}$, and $t'_1 := \chi_B \subset \mathbf{R}^{[0,1]}$. Put $E := \operatorname{lin}(T \cup \{t'_1\}) \subset \mathbf{R}^{[0,1]}$ endowed with the topology induced by the product-topology τ_p of $\mathbf{R}^{[0,1]}$. We note that $\operatorname{lin}(T)$ and E are dense in $\mathbf{R}^{[0,1]}$ w.r.t. τ_p, which yields $E' = \oplus_{x\in[0,1]} \mathbf{R}_x$ (where $\mathbf{R}_x: = \mathbf{R}$), the direct sum of the $\mathbf{R}_x$.

Let a l.s.p. $(X_t)_{t\in E}$ over the probability space $([0,1], \mathfrak{A}(\mathfrak{B} \cup \{B\}), \tilde{\lambda})$ be defined by $X_t(x) := t(x)\ \forall t \in E, \forall x \in [0,1]$.

ad(a): It is straightforward checking σ-smoothness at $\varnothing$ of the cylindrical measure $\mu_{0,X}$. Thus $\mu_{0,X}|\mathfrak{Z}_{E',E}$ has a unique extension to a σ-additive measure $\mu_X|\mathfrak{A}(\mathfrak{Z}_{E',E})$. Since E has countable dimension, E'_σ is metrizable and separable; therefore E'_σ has countable base for the topology which yields $\mathfrak{A}(\mathfrak{Z}_{E',E}) = \mathfrak{B}(E'_\sigma)$ and τ-smoothness of every measure on $\mathfrak{B}(E'_\sigma)$. This proves (a).

ad(b): We have to show that $\mu_X|\mathfrak{B}(E'_\sigma)$ is not tight. We define $\Delta :=$ $\{\delta_x: x \in [0,1]\} \subset E'$ where $\langle \delta_x, t\rangle := t(x)\ \forall t \in E$. It is checked quickly that $\mu_X(A) = \tilde{\lambda}(\{x \in [0,1]: \delta_x \in A\})\ \forall A \in \mathfrak{B}(E'_\sigma)$. Therefore, for $B' := \{f \in E': \langle f, t'_1\rangle = 1\}$, $\mu_X(B') = \tilde{\lambda}(\{x \in [0,1]: t'_1(x) = 1\}) = \tilde{\lambda}(B) = 1$, and it suffices to be shown:

$$(*) \qquad \mu_X(K) = 0 \quad \forall K \in \mathfrak{K}(E'_\sigma) \text{ such that } K \subset B'.$$

To prove $(*)$ we first show

$$(**) \qquad \Delta = \overline{\Delta}^\sigma.$$

PROOF OF $(**)$. Consider $f \in \overline{\Delta}^\sigma$; we get

$$U_{\varepsilon,S} := \{x \in [0,1]: |\langle \delta_x, t_i\rangle - \langle f, t_i\rangle| \leqslant \varepsilon\ \forall i \in \{1,\ldots,n\}\} \neq \varnothing$$
$$\forall \varepsilon > 0, \forall S = (t_1,\ldots,t_n) \in \mathfrak{S}_{\operatorname{lin}(T)}.$$

Since the elements of $\operatorname{lin}(T)$ are continuous on $[0,1]$, the $U_{\varepsilon,S}$ form a system of compact subsets of $[0,1]$ that is filtering downward. Thus $\bigcap_{\varepsilon\geqslant 0; S\in\mathfrak{S}_{\operatorname{lin}(T)}} U_{\varepsilon,S} \neq \varnothing$, i.e. there is $x_0 \in [0,1]$ such that $\langle \delta_{x_0}, t\rangle = \langle f, t\rangle\ \forall t \in \operatorname{lin}(T)$. Furthermore, $f = \delta_{x_0}$, as $\operatorname{lin}(T)$ is dense in E w.r.t. τ_p. This proves $(**)$. Next we need

$$(***)\qquad \begin{array}{l}\Phi\colon \Delta \to [0,1],\ \delta_x \to x, \text{ is continuous, if}\\ \Delta \subset E' \text{ is endowed with the topology induced by } E'_\sigma.\end{array}$$

PROOF OF $(***)$. We show: $\Phi^{-1}(S_\varepsilon(y)) \subset \Delta$ open, where $S_\varepsilon(y) := [0,1] \cap]y-\varepsilon, y+\varepsilon[$, $\varepsilon > 0$, $y \in [0,1]$. Consider $\delta_{y'} \in \Phi^{-1}(S_\varepsilon(y))$, i.e. $y' \in S_\varepsilon(y)$. As is easily seen, there is $k = k(y') \geqslant 2$ such that $t'_k(y') > 2^{-1}$ and the support of t'_k is contained in $S_\varepsilon(y)$. Thus we get

$$\delta_{y'} \in \{\delta_x \in \Delta\colon |\langle \delta_x, t'_k\rangle - \langle \delta_{y'}, t'_k\rangle| < 2^{-1}\} \subset \Phi^{-1}(S_\varepsilon(y)).$$

This implies that $\Phi^{-1}(S_\varepsilon(y))$ is open and proves $(***)$.

We now consider $K \in \mathcal{K}(E'_\sigma)$, $K \subset B'$. By $(**)$, $K \cap \Delta$ is a compact subset of Δ. By $(***)$ we get $\{x \in [0,1]\colon \delta_x \in K\} = \{x \in [0,1]\colon \delta_x \in K \cap \Delta\} \subset [0,1]$ compact, therefore $\{x \in [0,1]\colon \delta_x \in K\} \in \mathfrak{B}$. Thus,

$$\begin{aligned}\mu_X(K) &= \tilde{\lambda}(\{x \in [0,1]\colon \delta_x \in K\}) = \lambda(\{x \in [0,1]\colon \delta_x \in K\})\\ &\leqslant \lambda_*(\{x \in [0,1]\colon \delta_x \in B'\}) = \lambda_*(B) = 0 \quad \forall K \in \mathcal{K}(E'_\sigma),\ K \subset B'.\end{aligned}$$

This proves $(*)$, and we are through.

(3.2) EXAMPLE. There is a l.c.s. E and a l.s.p. $(X_t)_{t\in E}$ such that the following hold true:

(a) $(X_t)_{t\in E}$ is realizable in E',

(b) $(X_t)_{t\in E}$ is not strongly realizable in E'.

PROOF. Let X be the set of all countable ordinals, endowed with the order topology. Consider $E := \mathcal{C}(X) := \{f\colon X \to \mathbf{R} \mid f \text{ continuous}\}$, endowed with the topology of compact convergence. Of course E' is the set of Radon-measures on X with compact support. It is well known that for every $t \in E$ there is a $\beta = \beta(t) \in X$ such that $\forall \alpha \geqslant \beta$: $t(\alpha) = t(\beta) =: \langle g, t\rangle$. Obviously, $g \in E^*$. Moreover, $g \notin E'$ (this follows from the fact that for every compact $K \subset X$ there is $t = t(K) \in \mathcal{C}(X)$ such that $t(\alpha) = 0\ \forall \alpha \in K$ and $\langle g, t\rangle = 1$. This, in turn, holds true because X is completely regular and every compact $K \subset X$ is bounded w.r.t. the order in X).

We now define a l.s.p. $(X_t)_{t\in E}$ by $X_t(\omega) := \langle g, t\rangle\ \forall t \in E$, $\forall \omega \in \Omega$.

ad(a): (a) is proved in [4, Exp. no. 1, Ex. 1].

ad(b): Suppose $\mu_{0,X}$ extends to a τ-smooth measure $\mu|\mathfrak{B}(E'_\sigma)$. Then, for every $S = (t_1, \ldots, t_n) \in \mathfrak{S}_E$:

$$(*)\qquad \begin{array}{l}\mu(F_S) = \mu_{0,X}(F_S) = 1\\ \qquad\qquad \text{for } F_S := \{f \in E'\colon \langle f, t_i\rangle = \langle g, t_i\rangle \quad \forall i \in \{1, \ldots, n\}\}.\end{array}$$

On the other hand, $g \notin E'$ implies $\bigcap_{S\in\mathfrak{S}_E} F_S = \varnothing$, and by τ-smoothness we get

$$\inf_{S \in \mathfrak{S}_E} \mu(F_S) = \mu\Big(\bigcap_{S \in \mathfrak{S}_E} F_S\Big) = \mu(\varnothing) = 0,$$

which contradicts (*).

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