

ON PERTURBATION OF UNSTABLE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In connection with a conjecture of J. M. Bownds, conditions will be given on the fundamental system of the solutions of the unstable differential equation $y'' + a(t)y = 0$ which assure that the differential equation $x'' + a(t)x = g(t, x, x')$ has a solution with the property

$$\limsup_{t \rightarrow \infty} (|x(t)| + |x'(t)|) = \infty \quad \text{as } t \rightarrow \infty,$$

provided that $g(t, x, x')$ is "sufficiently small".

1. J. M. Bownds recently showed that from some theorems of stability theory it follows that the differential equation

$$(1) \quad y'' + a(t)y = 0, \quad t \geq 0,$$

for arbitrary $a(t) \in C[0, \infty)$ has a solution $y(t)$ with property

$$(R_1) \quad \limsup_{t \rightarrow \infty} (|y(t)| + |y'(t)|) > 0.$$

He also established that the equation

$$(2) \quad x'' + a(t)x = g(t, x, x'), \quad t \geq 0,$$

has the above property too, provided that the zero solution $y = 0$ of (1) is stable and there exists a function $\gamma(t) \in L[0, \infty)$ such that

$$(3) \quad |g(t, x, x')| \leq \gamma(t)(|x| + |x'|)$$

for $(t, x, x') \in [0, \infty) \times R \times R$.

As it was remarked in [1], it seems reasonable to conjecture that the latter statement is still true if the zero solution $y = 0$ of (1) is unstable.

It is well known that a necessary and sufficient condition for the zero solution of (1) to be unstable is the existence of a solution $y(t)$ with

$$(R_2) \quad \limsup_{t \rightarrow \infty} (|y(t)| + |y'(t)|) = \infty.$$

In this article we prove that the conjecture of J. M. Bownds holds under certain conditions; moreover in this case there exists a solution of (2) with property (R_2) .

2. Let us denote by R^2 the space of column vectors $z = \text{col}(z_1, z_2) = (z_1, z_2)^*$ and define the norm of z by $\|z\| = |z_1| + |z_2|$. We shall denote by $y_1(t), y_2(t)$ the solutions of (1) with the initial conditions $y_1(0) = 1, y'_1(0) = 0$,

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$y_2(0) = 0, y'_2(0) = 1$ and set

$$Y(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix}.$$

THEOREM. Suppose that the zero solution $y = 0$ of (1) is unstable and there exist two projections P_1, P_2 on the plane and a constant K such that $P_1 + P_2 = I$ and

$$(4) \quad \|Y(t)P_1(-y_2(s), y_1(s))^*\| \leq K, \quad 0 \leq s \leq t < \infty,$$

$$(5) \quad \|Y(t)P_2(-y_2(s), y_1(s))^*\| \leq K, \quad 0 \leq t \leq s < \infty.$$

Assume further that there exists a function $\gamma(t) \in L[0, \infty)$ such that

$$(3') \quad |g(t, x_1, x'_1) - g(t, x_2, x'_2)| \leq \gamma(t)(|x_1 - x_2| + |x'_1 - x'_2|)$$

for $(t, x_1, x'_1), (t, x_2, x'_2) \in [0, \infty) \times R \times R$.

Then equation (2) has a solution $x(t)$ with property (R_2) .

PROOF. Since $Y^{-1}(s)(0, z_2)^* = z_2(-y_2(s), y_1(s))^*$, from (4) and (5) we have

$$(4') \quad \|Y(t)P_1 Y^{-1}(s)(0, z_2)^*\| \leq K|z_2|, \quad 0 \leq s \leq t < \infty,$$

$$(5') \quad \|Y(t)P_2 Y^{-1}(s)(0, z_2)^*\| \leq K|z_2|, \quad 0 \leq t \leq s < \infty$$

for all z_2 .

Taking into account the special form of (2), by a slight modification¹ of Theorem 11 of [2, p. 76] we obtain, assuming (4'), (5') and (3'), that there exists a one-to-one correspondence between the $\|\cdot\|$ -bounded solutions $y(t)$ of (1) and $x(t)$ of (2).

Let t_1 be a sufficiently large fixed number; then this correspondence satisfies

$$(6) \quad P_1 Y^{-1}(t_1)(x(t_1), x'(t_1))^* = P_1 Y^{-1}(t_1)(y(t_1), y'(t_1))^*$$

and is bicontinuous in the space $C([t_1, \infty); R^2)$ with the supremum norm.

First consider the case $P_1 \neq 0$. From (4) it follows that the function $\|Y(t)P_1\|$ is bounded; consequently every solution $y(t)$ of (1) with the property $Y^{-1}(t_1)(y(t_1), y'(t_1))^* \in P_1 R^2$ is $\|\cdot\|$ -bounded. Moreover, from the instability of $y = 0$ we obtain that all $\|\cdot\|$ -bounded solutions have this property, i.e., $P_1 \neq I$.

Therefore on the straight line $P_1 Y^{-1}(t_1)(y, y')^* = P_1(c, c')^*$ (c and c' are constants), there is a unique point, through which a $\|\cdot\|$ -bounded solution of (1) is passing at $t = t_1$. Because of (6) the same is true also for equation (2). Consequently, equation (2) has a solution satisfying (R_2) .

Now suppose that $P_1 = 0$, i.e., $P_2 = I$. If every nontrivial solution of (1) is

¹We modify Theorem 11 of [2] by replacing Coppel's condition (23) by

$$(23') \quad \|Y(t)P_1 Y^{-1}(s)P\| \leq K \quad \text{for } t_0 < s \leq t,$$

$$\|Y(t)P_2 Y^{-1}(s)P\| \leq K \quad \text{for } t_0 < t \leq s.$$

Here P is a projection such that $Pf(t, x) = f(t, x)$. In our case if $x = (x_1, x'_1)^* \in R^2$, then $f(t, x) = (0, g(t, x_1, x'_1))^*$.

not $\|\cdot\|$ -bounded, then from the above-mentioned theorem it follows that every solution of (2) satisfies (R_2) . If (1) has $\|\cdot\|$ -bounded solutions, then these are passing through the points of a straight line e of R^2 at $t = t_1$ (otherwise (1) would have two linearly independent $\|\cdot\|$ -bounded solutions, which contradicts the hypothesis that the solution $y = 0$ is unstable) and are conditionally stable (see [2, p. 76]) with respect to the points of the straight line e ; that is, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $(y_1(t_1), y'_1(t_1))^* \in e$, $(y_2(t_1), y'_2(t_1))^* \in e$, $|y_1(t_1) - y_2(t_1)| + |y'_1(t_1) - y'_2(t_1)| < \delta$, then $|y_1(t) - y_2(t)| + |y'_1(t) - y'_2(t)| < \epsilon$ for every $t \geq t_1$. From this fact it follows that the mapping $e \rightarrow C([t_1, \infty); R^2)$, defined by the correspondence $(y(t_1), y'(t_1))^* \rightarrow (y(t), y'(t))^*$, is one-to-one and continuous. On the other hand, according to the theorem cited above, the set of $\|\cdot\|$ -bounded solutions $y(t)$ of (1) can be continuously and in a one-to-one manner mapped onto the set of $\|\cdot\|$ -bounded solutions $x(t)$ of equation (2). Furthermore, the mapping from $C([t_1, \infty); R^2)$ onto R^2 defined by $(x(t), x'(t))^* \rightarrow (x(t_1), x'(t_1))^*$ is obviously one-to-one and continuous. If equation (2) would not have a solution with property (R_2) , then the mapping $e \rightarrow R^2$: $(y(t_1), y'(t_1))^* \rightarrow (x(t_1), x'(t_1))^*$ would be a one-to-one and continuous mapping of e onto R^2 , which is impossible. Our theorem is proved.

REMARK. Conditions (4) and (5) can certainly be satisfied if $a(t) = a \neq 0$ (a is a constant), or $a(t)$ is a periodic function with period τ and $(y_1(\tau) + y_2(\tau))^2 \neq 4$ (see [2, pp. 48, 78]).

REFERENCES

1. J. M. Bownds, *Stability implications on the asymptotic behavior of second order differential equations*, Proc. Amer. Math. Soc. **39** (1973), 169–172. MR **47** #2150.
2. W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath, Boston, Mass., 1965. MR **32** #7875.

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