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ON PERTURBATION OF UNSTABLE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In connection with a conjecture of J. M. Bownds, conditions will be given on the fundamental system of the solutions of the unstable differential equation $y'' + a(t)y = 0$ which assure that the differential equation $x'' + a(t)x = g(t, x, x')$ has a solution with the property

$$\limsup(|x(t)| + |x'(t)|) = \infty \quad \text{as } t \to \infty,$$

provided that $g(t, x, x')$ is "sufficiently small".

1. J. M. Bownds recently showed that from some theorems of stability theory it follows that the differential equation

$$y'' + a(t)y = 0, \qquad t \geqslant 0, \tag{1}$$

for arbitrary $a(t) \in C[0, \infty)$ has a solution $y(t)$ with property

$$\limsup_{t\to\infty}(|y(t)| + |y'(t)|) > 0. \tag{R_1}$$

He also established that the equation

$$x'' + a(t)x = g(t, x, x'), \qquad t \geqslant 0, \tag{2}$$

has the above property too, provided that the zero solution $y = 0$ of (1) is stable and there exists a function $\gamma(t) \in L[0, \infty)$ such that

$$|g(t, x, x')| \leqslant \gamma(t)(|x| + |x'|) \tag{3}$$

for $(t, x, x') \in [0, \infty) \times R \times R$.

As it was remarked in [1], it seems reasonable to conjecture that the latter statement is still true if the zero solution $y = 0$ of (1) is unstable.

It is well known that a necessary and sufficient condition for the zero solution of (1) to be unstable is the existence of a solution $y(t)$ with

$$\limsup_{t\to\infty}(|y(t)| + |y'(t)|) = \infty. \tag{R_2}$$

In this article we prove that the conjecture of J. M. Bownds holds under certain conditions; moreover in this case there exists a solution of (2) with property (R_2).

2. Let us denote by R^2 the space of column vectors $z = \mathrm{col}(z_1, z_2) = (z_1, z_2)^*$ and define the norm of z by $\|z\| = |z_1| + |z_2|$. We shall denote by $y_1(t), y_2(t)$ the solutions of (1) with the initial conditions $y_1(0) = 1, y_1'(0) = 0$,

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$y_2(0) = 0, y_2'(0) = 1$ and set

$$Y(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}.$$

THEOREM. *Suppose that the zero solution* $y = 0$ *of* (1) *is unstable and there exist two projections* P_1, P_2 *on the plane and a constant* K *such that* $P_1 + P_2 = I$ *and*

$$\|Y(t)P_1(-y_2(s), y_1(s))^*\| \leqslant K, \qquad 0 \leqslant s \leqslant t < \infty, \tag{4}$$

$$\|Y(t)P_2(-y_2(s), y_1(s))^*\| \leqslant K, \qquad 0 \leqslant t \leqslant s < \infty. \tag{5}$$

Assume further that there exists a function $\gamma(t) \in L[0, \infty)$ *such that*

$$|g(t, x_1, x_1') - g(t, x_2, x_2')| \leqslant \gamma(t)(|x_1 - x_2| + |x_1' - x_2'|) \tag{3'}$$

for $(t, x_1, x_1'), (t, x_2, x_2') \in [0, \infty) \times R \times R$.

Then equation (2) *has a solution* $x(t)$ *with property* (R_2).

PROOF. Since $Y^{-1}(s)(0, z_2)^* = z_2(-y_2(s), y_1(s))^*$, from (4) and (5) we have

$$\|Y(t)P_1Y^{-1}(s)(0, z_2)^*\| \leqslant K|z_2|, \qquad 0 \leqslant s \leqslant t < \infty, \tag{4'}$$

$$\|Y(t)P_2Y^{-1}(s)(0, z_2)^*\| \leqslant K|z_2|, \qquad 0 \leqslant t \leqslant s < \infty \tag{5'}$$

for all z_2.

Taking into account the special form of (2), by a slight modification[1] of Theorem 11 of [2, p. 76] we obtain, assuming (4′), (5′) and (3′), that there exists a one-to-one correspondence between the $\|\cdot\|$-bounded solutions $y(t)$ of (1) and $x(t)$ of (2).

Let t_1 be a sufficiently large fixed number; then this correspondence satisfies

$$P_1Y^{-1}(t_1)(x(t_1), x'(t_1))^* = P_1Y^{-1}(t_1)(y(t_1), y'(t_1))^* \tag{6}$$

and is bicontinuous in the space $C([t_1, \infty); R^2)$ with the supremum norm.

First consider the case $P_1 \neq 0$. From (4) it follows that the function $\|Y(t)P_1\|$ is bounded; consequently every solution $y(t)$ of (1) with the property $Y^{-1}(t_1)(y(t_1), y'(t_1))^* \in P_1R^2$ is $\|\cdot\|$-bounded. Moreover, from the instability of $y = 0$ we obtain that all $\|\cdot\|$-bounded solutions have this property, i.e., $P_1 \neq I$.

Therefore on the straight line $P_1Y^{-1}(t_1)(y, y')^* = P_1(c, c')^*$ (c and c' are constants), there is a unique point, through which a $\|\cdot\|$-bounded solution of (1) is passing at $t = t_1$. Because of (6) the same is true also for equation (2). Consequently, equation (2) has a solution satisfying (R_2).

Now suppose that $P_1 = 0$, i.e., $P_2 = I$. If every nontrivial solution of (1) is

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[1] We modify Theorem 11 of [2] by replacing Coppel's condition (23) by

$$\begin{aligned} &\|Y(t)P_1Y^{-1}(s)P\| \leqslant K \quad \text{for } t_0 \leqslant s \leqslant t, \\ &\|Y(t)P_2Y^{-1}(s)P\| \leqslant K \quad \text{for } t_0 \leqslant t \leqslant s. \end{aligned} \tag{23'}$$

Here P is a projection such that $Pf(t, x) = f(t, x)$. In our case if $x = (x_1, x_1')^* \in R^2$, then $f(t, x) = (0, g(t, x_1, x_1'))^*$.

not $\|\cdot\|$-bounded, then from the above-mentioned theorem it follows that every solution of (2) satisfies (R_2). If (1) has $\|\cdot\|$-bounded solutions, then these are passing through the points of a straight line $\mathbf{e}$ of R^2 at $t = t_1$ (otherwise (1) would have two linearly independent $\|\cdot\|$-bounded solutions, which contradicts the hypothesis that the solution $y = 0$ is unstable) and are conditionally stable (see [2, p. 76]) with respect to the points of the straight line $\mathbf{e}$; that is, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $(y_1(t_1), y_1'(t_1))^* \in \mathbf{e}$, $(y_2(t_1), y_2'(t_1))^* \in \mathbf{e}$, $|y_1(t_1) - y_2(t_1)| + |y_1'(t_1) - y_2'(t_1)| < \delta$, then $|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)| < \varepsilon$ for every $t \geqslant t_1$. From this fact it follows that the mapping $\mathbf{e} \to C([t_1, \infty); R^2)$, defined by the correspondence $(y(t_1), y'(t_1))^* \to (y(t), y'(t))^*$, is one-to-one and continuous. On the other hand, according to the theorem cited above, the set of $\|\cdot\|$-bounded solutions $y(t)$ of (1) can be continuously and in a one-to-one manner mapped onto the set of $\|\cdot\|$-bounded solutions $x(t)$ of equation (2). Furthermore, the mapping from $C([t_1, \infty); R^2)$ onto R^2 defined by $(x(t), x'(t))^* \to (x(t_1), x'(t_1))^*$ is obviously one-to-one and continuous. If equation (2) would not have a solution with property (R_2), then the mapping $\mathbf{e} \to R^2$: $(y(t_1), y'(t_1))^* \to (x(t_1), x'(t_1))^*$ would be a one-to-one and continuous mapping of $\mathbf{e}$ onto R^2, which is impossible. Our theorem is proved.

REMARK. Conditions (4) and (5) can certainly be satisfied if $a(t) = a \neq 0$ (a is a constant), or $a(t)$ is a periodic function with period τ and $(y_1(\tau) + y_2(\tau))^2 \neq 4$ (see [2, pp. 48, 78]).

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