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CHARACTERISTIC NUMBERS FOR UNORIENTED SINGULAR G-BORDISM

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ABSTRACT. We develop the notion of characteristic numbers for unoriented singular G-manifolds in a G-space, G being a finite group, and prove their invariance with respect to unoriented singular G-bordism.

Thom [5] gave the notion of Stiefel Whitney numbers and Pontrjagin numbers of a manifold M^n and proved its invariance with respect to bordism. Chung N. Lee and Arthur Wasserman [4] developed these notions for G-manifolds. In this note we have developed these notions for unoriented singular principal G-manifolds in a G-space, G being a finite group, and proved their invariance with regard to unoriented singular G-bordism.

1. **Characteristic numbers.** Let X be a finite CW-complex with free action of G, G being a finite group, and X/G be again a finite CW-complex. Let h^* be an equivariant cohomology theory and h_* be the associated equivariant homology theory [1]. Let $h^* = H^* \circ A$ and $h_* = H_* \circ A$, where A is a functor from the category of G-spaces and equivariant maps to the category of topological spaces and continuous maps, H^* is the singular cohomology theory and H_* is the associated singular homology theory. Let

$$\langle\ ,\ \rangle\colon h^*(X;G) \otimes_{H^*(\mathrm{pt.})} h_*(X;G) \to H_*(\mathrm{pt.})$$

be the Kronecker pairing.

Let us assign to each compact G-manifold W, a class

$$[W, \partial W] \in h_*(W, \partial W; G)$$

such that

(a) $[W_1 \cup W_2, \partial W_1 \cup \partial W_2] = [W_1, \partial W_1] + [W_2, \partial W_2]$,

(b) $\partial_*[W, \partial W] = [\partial W]$.

Suppose $[M^n, f; G]$ is an element of unoriented bordism group $\mathfrak{N}_n(X; G)$ [3] and $x \in h^*(B(O,G)_n; G), B(O,G)_n$ being the classifying space for G-vector bundles of dimension n. Then the x-characteristic number of the map $f\colon M^n \to X$ associated with an element $a^m \in h^m(X;G)$ is defined to be

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$\langle \tau^*_{M^n}(x) f^*(a^m), [M] \rangle$, where $\tau_{M^n}\colon M^n \to B(O,G)_n$ is the tangent map.

In particular, let the equivariant cohomology h^* be given by $h^*(X;G) = H^*((E_G \times X)/G; \mathbf{Z}_2)$ and h_* be the associated equivariant homology, i.e. $h_*(X;G) = H_*((E_G \times X)/G; \mathbf{Z}_2)$, where the action of G on $E_G \times X$ is given by $g(e,x) = (ge, gx)$, E_G being the total space of the universal G-bundle. Consider the map $q\colon X/G \to (E_G \times X)/G$ given by $q([x]) = [\overline{\alpha}(x), x]$, where $\overline{\alpha}$ is the map given by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\overline{\alpha}} & E_G \\ \downarrow & & \downarrow \\ X/G & \xrightarrow{\alpha} & BG \end{array}$$

The map q is homotopy equivalence. Thus

$$h^*(X;G) \overset{q^*}{\approx} H^*(X/G; \mathbf{Z}_2) \quad \text{and} \quad h_*(X;G) \overset{q_*^{-1}}{\approx} H_*(X/G; \mathbf{Z}_2).$$

Therefore $h_*(M^n;G) \approx H_*(M^n/G; \mathbf{Z}_2)$ has a topological class, say σ_n, in dimension n.

2. **Invariance of characteristic numbers.** Throughout the section we will be considering equivariant cohomology h^* to be

$$h^*(X;G) = H^*((E_G \times X)/G; \mathbf{Z}_2)$$

and equivariant homology h_* to be

$$h_*(X;G) = H_*((E_G \times X)/G; \mathbf{Z}_2).$$

THEOREM 2.1. *If $[M^n, f; G] \in \mathfrak{N}_n(X;G)$ is zero then all the x-characteristic numbers of the map $f\colon M^n \to X$ associated with every $a^m \in h^m(X;G)$ are zero.*

PROOF. Since $[M^n, f; G] \in \mathfrak{N}_n(X;G)$ is zero, $\exists$ an $(n+1)$-dimensional compact principal G-manifold W^{n+1} and an equivariant map $F\colon W^{n+1} \to X$ with $\partial W^{n+1} = M^n$ and $F/M^n = f$. Let $\omega_{n+1} \in h_{n+1}(W^{n+1}, \partial W^{n+1}; G)$ be the topological class of W^{n+1}. Then $\partial_*(\omega_{n+1}) = \sigma_n$. We have the following commutative diagram:

$$\begin{array}{ccc} h^*(B(O,G)_n;G) & \xrightarrow{\tau^*_{M^n}} & h^*(M^N;G) \\ \uparrow j^* & & \uparrow i^* \\ h^*(B(O,G)_{n+1};G) & \xrightarrow{\tau^*_{W^{n+1}}} & h^*(W^{n+1};G), \end{array}$$

where $j\colon B(O,G)_n \to B(O,G)_{n+1}$ is the map classifying $\mu_n \oplus 1$, $\mu_n \to B(O,G)_n$ being the universal G-vector bundle. Also we have

$$\begin{aligned} h^*(B(O,G)_n;G) &= H^*((E_G \times B(O,G)_n)/G; \mathbf{Z}_2) \\ &= H^*(BG \times BO_n; \mathbf{Z}_2) \quad [6] \\ &= H^*(BG; \mathbf{Z}_2) \otimes H^*(BO_n; \mathbf{Z}_2) \end{aligned}$$

and

$$h_*(B(O,G)_n;G) = H_*(BG;\mathbf{Z}_2) \otimes H_*(BO_n;\mathbf{Z}_2).$$

Thus the map j^* is a surjection. Therefore for every $x \in h^*(B(O,G)_n;G)$, $\exists y \in h^*(B(O,G)_{n+1};G)$ such that $j^*(y) = x$. Therefore

$$\begin{aligned}\langle \tau^*_{M^n}(x) f^*(a^m), \sigma_n\rangle &= \langle \tau^*_{M^n} j^*(y) f^*(a^m), \sigma_n\rangle = \langle i^* \tau^*_{W^{n+1}}(y) i^* F^*(a^m), \sigma_n\rangle \\ &= \langle \tau^*_{W^{n+1}}(y) F^*(a^m), i_* \partial_*(\omega_{n+1})\rangle = 0.\end{aligned}$$

This completes the proof of the theorem.

Consider now the map $\mu : \mathfrak{N}_*(X;G) \to h_*(X;G)$ defined as $\mu([M^n,f;G]) = q_* \bar{f}_*(\bar{\sigma}_n)$, where $\bar{f}$ is the map given by the following commutative diagram:

$$\begin{array}{ccc} M^n & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ M^n/G & \xrightarrow{\bar{f}} & X/G \end{array}$$

$\bar{\sigma}_n \in H_n(M^n/G;\mathbf{Z}_2)$ being the fundamental class and let $\bar{\mu}$: $\mathfrak{N}_*(X/G) \to H_*(X/G;\mathbf{Z}_2)$ be the map defined by $\bar{\mu}([N^n,g]) = g_*(\bar{\sigma}'_n)$, where $\bar{\sigma}'_n \in H_n(N^n;\mathbf{Z}_2)$ is the fundamental class. Suppose $\phi_* : \mathfrak{N}_*(X;G) \to \mathfrak{N}_*(X/G)$ is the isomorphism [3] defined as $\phi_*([M^n,f;G]) = [M^n/G,\bar{f}]$. Then $\mu = q_* \bar{\mu}\phi_*$ and, therefore, μ is an epimorphism, since $\bar{\mu}$ is so [2]. For every $a \in h_*(X;G)$, we select $[M^n,f;G] \in \mathfrak{N}_*(X;G)$ such that $\mu([M^n,f;G]) = a$. We define the $\mathfrak{N}$-module structure on $h_*(X;G)$ by

$$[V^m]a = \mu[M^n \times V^m, f'; G],$$

for every $[V^m] \in \mathfrak{N}$, where the action of G on $M^n \times V^m$ is defined as $g(x,y) = (gx,y)$ and f': $M^n \times V^m \to X$ is defined as $f'(x,y) = f(x)$. Thus $h_*(X;G) \otimes \mathfrak{N}$ is a $\mathfrak{N}$-module. Let $\{C_{n,i}\}$ be the additive base of $h_*(X;G)$. Let $[M_i^n,f_i;G] \in \mathfrak{N}_*(X;G)$ with $\mu([M_i^n,f_i;G]) = C_{n,i}$. We define h: $h_*(X;G) \otimes \mathfrak{N} \to \mathfrak{N}_*(X;G)$ by $h(C_{n,i} \otimes 1) = [M_i^n,f_i;G]$.

THEOREM 2.2. *The map h: $h_*(X;G) \otimes \mathfrak{N} \to \mathfrak{N}_*(X;G)$, defined as above is an isomorphism.*

PROOF. We have the following commutative diagram:

$$\begin{array}{ccc} h_*(X;G) \otimes \mathfrak{N} & \xrightarrow{h} & \mathfrak{N}_*(X;G) \\ \downarrow q_*^{-1} \otimes 1_{\mathfrak{N}} & & \downarrow \phi_* \\ H_*(X/G;\mathbf{Z}_2) \otimes \mathfrak{N} & \xrightarrow{\bar{h}} & \mathfrak{N}_*(X/G) \end{array}$$

where $\bar{h}$: $H_*(X/G;\mathbf{Z}_2) \otimes \mathfrak{N} \to \mathfrak{N}_*(X/G)$ is defined as $\bar{h}(\bar{C}_{n,i} \otimes 1) = [M_i^n/G; \bar{f}_i]$, where $\bar{C}_{n,i} = q_*^{-1}(C_{n,i})$. We already know that $\bar{h}$ is an isomorphism [2] and, therefore, so is h.

The above theorem gives the converse of Theorem 2.1 given as below.

THEOREM 2.3. *If all the characteristic numbers of an element* $[M^n, f; G] \in \mathfrak{N}_*(X; G)$ *are zero, then* $[M^n, f; G] = 0$.

PROOF. Let $\mu([M^n, f; G]) = C_n \in h_*(X; G)$ and $q_*^{-1}(C_n) = \overline{C}_n \in H_*(X/G; \mathbf{Z}_2)$. Therefore $\bar{f}_*(\bar{\sigma}_n) = \overline{C}_n$. Suppose $\{C_{n,i}\}_{i \in I}$ is an additive base of $h_n(X; G)$ and $C^{n,j} \in h^n(X; G)$ is the cohomology class dual to $C_{n,i}$ in the sense $\langle \overline{C}^{n,j}, \overline{C}_{n,i} \rangle = \delta_{ij}$, where $q_*^{-1}(C_{n,i}) = \overline{C}_{n,i}$ and $q^*(C^{n,j}) = \overline{C}^{n,j}$. Let $C_n = \sum_{i \in S} \pm C_{n,i}$, S being a finite subset of I. Then if $C^n = \sum_{i \in S} \pm C^{n,i}$, by hypothesis the x-characteristic number of $[M^n, f; G]$ associated with $C^n \in h^n(X; G)$ is zero, that means taking x to be unit class of $h^*(B(O, G)_n; G)$,

$$\langle f^*(C^n), [M] \rangle = 0, \quad \text{or} \quad \langle f^*(C^n), q_*[\bar{\sigma}_n] \rangle = 0,$$

or

$$\langle (q^*)^{-1}(\overline{C}^n), f_* q_*[\bar{\sigma}_n] \rangle = 0, \quad \text{or} \quad \langle (q^*)^{-1}(\overline{C}^n), q_* \bar{f}_*[\bar{\sigma}_n] \rangle = 0,$$

by the following commutative diagram

$$\begin{array}{ccc} h_n(M^n; G) & \xrightarrow{f_*} & h_n(X; G) \\ \downarrow q_*^{-1} & & \downarrow q_*^{-1} \\ H_n(M^n/G; \mathbf{Z}_2) & \xrightarrow{\bar{f}_*} & H_n(X/G; \mathbf{Z}_2) \end{array}$$

Therefore $\langle (q^*)^{-1}(\overline{C}^n), q_*(\overline{C}_n) \rangle = 0$, which implies that $\langle \overline{C}^n, \overline{C}_n \rangle = 0$, showing that $\overline{C}_n = 0$. Also it is easy to see that $h(C_n \otimes 1) = [M^n, f; G]$. Since h is an isomorphism and $C_n = 0$, $[M^n, f; G] = 0$, which completes the proof of the theorem.

Theorems 2.1 and 2.3 give the invariance of characteristic numbers with regard to unoriented singular principal G-bordism.

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