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NONIMMERSIONS OF FLAG MANIFOLDS

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ABSTRACT. Certain useful quadratic identities in the cohomology of classifying spaces induce quadratic equations in the cohomology of a manifold M under the classifying map for the normal bundle of M. In low dimensional flag manifolds, one can show that the quadratic equation has no root, thus establishing a nonimmersion.

1. **Introduction.** In integral cohomology, one has that

$$p_r = \chi^2 \in H^{4r}(\mathrm{BSO}(2r); Z) \tag{1.1}$$

where BSO($2r$) is the classifying space for oriented real r-plane bundles, p_r is the rth Pontryagin class and χ is the Euler class. An oriented manifold M immerses in codimension k if and only if the stable normal bundle has dimension k. (Cf. [3].) We obtain nonimmersion by showing that the quadratic equation in $H^*(M; Z)$, induced by (1.1), has no integral solution.

If $n = n_1 + \cdots + n_q$, let

$$F(n_1, \ldots, n_q) = U(n)/U(n_1) \times \cdots \times U(n_q)$$

denote a complex incomplete flag manifold. We shall show that:

(a) If $M = F(2,4)$, then $\dim M = 16$, $M \subseteq R^{31}$, $\nsubseteq R^{24}$.

(b) If $M = F(2,5)$, then $\dim M = 20$, $M \subseteq R^{39}$, $\nsubseteq R^{30}$.

The following flag manifolds are bundles with fibre or base equal to $F(2,4)$ or $F(2,5)$ and so results (a) and (b) together with [8] imply the following results.

If $M = F(2,2,2)$, then $\dim M = 24$, $M \subseteq R^{33}$, $\nsubseteq R^{32}$.
If $M = F(2,1,3)$, then $\dim M = 22$, $M \subseteq R^{33}$, $\nsubseteq R^{30}$.
If $M = F(2,2,3)$, then $\dim M = 32$, $M \subseteq R^{46}$, $\nsubseteq R^{42}$.
If $M = F(2,3,2)$, then $\dim M = 32$, $M \subseteq R^{46}$, $\nsubseteq R^{42}$.
If $M = F(2,1,4)$, then $\dim M = 28$, $M \subseteq R^{46}$, $\nsubseteq R^{38}$.
If $M = F(1,2,4)$, then $\dim M = 28$, $M \subseteq R^{46}$, $\nsubseteq R^{36}$.
If $M = F(1,2,5)$, then $\dim M = 34$, $M \subseteq R^{61}$, $\nsubseteq R^{44}$.

An interesting aspect of the above results is that the questions of immersion and nonimmersion are now completely solved for the flag manifold $F(2,2,2)$.

The method of this paper was used by Connell [2] to obtain nonimmersions of $F(2,2)$, $F(2,3)$, $F(1,2,2)$ and $F(2,2,2)$. This paper gives an improvement on the result of Connell [2] in the case of the flag manifold $F(2,2,2)$.

It seems that the following conjecture is reasonable.

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CONJECTURE. If $M = F(2, n-1)$, then $\dim M = 4n-4$, $M \subseteq R^{8n-9}$, $\nsubseteq R^{6n-6}$.

2. $F(2,4)$. Let $\mathrm{GL}(n) \to F(q, n-q)$ be a principal $\mathrm{GL}(q, n-q)$-bundle, λ, where $\mathrm{GL}(n)$ is the general linear group with entries in the field of complex numbers and $\mathrm{GL}(q, n-q)$ is a subgroup of $\mathrm{GL}(n)$ consisting of all matrices of the form

$$A = \left[\begin{array}{c|c} A' & B \\ \hline 0 & A'' \end{array}\right],$$

where $A' \in \mathrm{GL}(q)$, $A'' \in \mathrm{GL}(n-q)$ and B is any $q \times (n-q)$-matrix. Let λ', λ'' be the subbundle and quotient-bundle associated with λ. Then λ' is a canonical q-plane bundle over $F(q, n-q)$. By the splitting principle, λ' splits into line bundles $\xi_0, \ldots, \xi_{q-1}$ such that the total Chern class of λ' is given by

$$c(\lambda') = \prod_{i=0}^{q-1} c(\xi_i) = \prod_{i=0}^{q-1} (1 + \gamma_i).$$

Thus

$$c_h(\lambda') = \sigma_h(\gamma_0, \ldots, \gamma_{q-1}), \qquad h = 1, \ldots, q,$$

where $\sigma_h = \sigma_h(\gamma_0, \ldots, \gamma_{q-1})$ denotes the hth elementary symmetric function in $\gamma_0, \ldots, \gamma_{q-1}$. We also denote by $\bar{\sigma}_h = \bar{\sigma}_h(\gamma_0, \ldots, \gamma_{q-1})$ the hth complete symmetric function in $\gamma_0, \ldots, \gamma_{q-1}$.

The Whitney sum $\lambda' \oplus \lambda''$ of the subbundle and quotient-bundle associated with λ is the trivial n-plane bundle over $F(q, n-q)$. Therefore, $c(\lambda')c(\lambda'') = 1$. Thus

$$c(\lambda'') = \sum_{i=0}^{\infty} (-1)^i \bar{\sigma}_i(\gamma_0, \ldots, \gamma_{q-1}),$$

and since λ'' is an $(n-q)$-plane bundle, then

$$\bar{\sigma}_i = \bar{\sigma}_i(\gamma_0, \ldots, \gamma_{q-1}) = 0, \qquad i > n-q.$$

We also, therefore, have that

$$\sum_{i=0}^{q} \sigma_i \sum_{j=0}^{\infty} (-1)^j \bar{\sigma}_j = 1$$

from which follow the relations:

$$\sum_{i=0}^{k} (-1)^i \sigma_i \bar{\sigma}_{k-i} = 0, \qquad k > 0. \tag{2.1}$$

Now, from [1, p. 522], we have that the σ_r $(1 \leqslant r \leqslant q)$ generate $H^*(F(q, n-q), Z)$ subject to the relations

$$\bar{\sigma}_{n-q+i} = 0, \qquad i \geqslant 0. \tag{2.2}$$

Also, from [9, §1.3] and [5, pp. 354–359], the Chern classes, c_i, of $F(2,4)$ are given by

$$\begin{aligned}
c_1 &= 6\sigma_1,\\
c_2 &= 16\bar{\sigma}_2 + 18\sigma_2,\\
c_3 &= 26\bar{\sigma}_3 + 58\sigma_2\sigma_1,\\
c_4 &= 31\bar{\sigma}_4 + 91\bar{\sigma}_2\sigma_2 + 67\sigma_2^2,\\
c_5 &= 90\sigma_2\bar{\sigma}_3 + 120\sigma_2^2\sigma_1,\\
c_6 &= 105\sigma_2^2\bar{\sigma}_2 + 65\sigma_2^3,\\
c_7 &= 60\sigma_2^3\sigma_1,\\
c_8 &= 15\sigma_2^4.
\end{aligned}$$

If $F(2,4)$ immerses in codimension 8, its stable normal bundle, ν, is an SO(8) bundle. If d_i denotes the ith Chern class of ν, then using the identities

$$\left(\sum_{i=0}^{8} c_i\right)\left(\sum_{i=0}^{8} d_i\right) = 1, \quad \sum_{i=0}^{k} c_{k-i}d_i = 0, \qquad k > 0,$$

and relations (2.1) and (2.2) when $q = 2$, $n = 6$, we obtain the following values of d_i, $1 \leqslant i \leqslant 8$.

$$\begin{aligned}
d_1 &= -6\sigma_1,\\
d_2 &= 20\bar{\sigma}_2 + 18\sigma_2,\\
d_3 &= -50\bar{\sigma}_3 - 82\sigma_2\sigma_1,\\
d_4 &= 105\bar{\sigma}_4 + 237\sigma_2\bar{\sigma}_2 + 129\sigma_2^2,\\
d_5 &= -546\sigma_2\bar{\sigma}_3 - 504\sigma_2^2\sigma_1,\\
d_6 &= 1323\sigma_2^2\bar{\sigma}_2 + 595\sigma_2^3,\\
d_7 &= -2100\sigma_2^3\sigma_1,\\
d_8 &= 2100\sigma_2^4.
\end{aligned}$$

Then from [4, Theorem 4.5.1], the Pontryagin class, $p_4(\nu)$, of ν can be computed from the formula

$$p_4(\nu) = 2d_8 - 2d_7d_1 + 2d_6d_2 - 2d_5d_3 + d_4^2$$

to give that $p_4(\nu) = -81\sigma_2^4$.

Now let $\chi = a\bar{\sigma}_4 + b\sigma_2\bar{\sigma}_2 + c\sigma_2^2$ be an element of $H^8(F(2,4); Z)$; then

$$\chi^2 = (a^2 + b^2 + c^2)\sigma_2^4.$$

Then identity (1.1) implies that $a^2 + b^2 + c^2 = -81$. This quadratic equation has no integral solution. Thus nonimmersion in codimension 8 is established. Therefore, together with [3], we have

PROPOSITION 2.3. *The sixteen dimensional manifold $F(2,4)$ does not immerse in codimension 8 but does immerse in codimension 15.*

3. $F(2,5)$.

PROPOSITION 3.1. *The twenty dimensional manifold* $F(2,5)$ *does not immerse in codimension* 10 *but does immerse in codimension* 19.

PROOF. The method of proof is completely the same as for Proposition 2.3 and the details are, therefore, omitted. Note that the Chern classes of $F(2,5)$ are contained in [6, Proposition 3.3] and if $F(2,5)$ immerses in codimension 10, one finds that the Pontryagin class $p_5(\nu)$ of its stable normal bundle ν is equal to $-504\sigma_2^5$.

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