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ON ANALYTIC FUNCTIONS INTO l^p-SPACES

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ABSTRACT. Let Δ be the open unit disc in the complex plane and let $1 \leqslant p < \infty$. We construct an example of a discontinuous function from Δ into l^p which is analytic as a function from Δ into $l^{p'}$ for any $p' > p$ and whose derivatives (into $l^{p'}$) of all orders are functions into l^p.

Throughout, we denote by Δ the open unit disc in the complex plane and by $\{e_i\}$ the canonical basis in the spaces l^p $(1 \leqslant p < \infty)$. Instead of $\sum_{i=1}^{\infty}$ we write $\sum$. We denote by N (resp. N_0) the set of all positive (resp. nonnegative) integers.

Suppose that $\zeta \mapsto f(\zeta) = \sum \varphi_i(\zeta)e_i = (\varphi_i(\zeta))$ is a function from Δ into l^p. In general the analyticity of all φ_i is not sufficient for analyticity of f, as shown by R. Aron and J. Cima [1]. They ask whether the analyticity of all φ_i together with uniform boundedness of φ_i implies analyticity of f.

In the paper we construct a discontinuous function from Δ into l^p $(1 \leqslant p < \infty)$ which is analytic on Δ as a function into $l^{p'}$ for any $p' > p$ and whose derivatives (into $l^{p'}$) of all orders are functions into l^p. In particular, this gives the negative answer to the question mentioned above.

LEMMA 1. *Let $1 \leqslant p < \infty$. There exists a sequence $\{\psi_i\}$ of complex-valued polynomials such that*

(i) $(\psi_i^{(n)}(\zeta)) \in l^p$ $(\zeta \in \Delta, n \in N_0)$;

(ii) *the sum of the series* $\sum |\psi_i(\zeta)|^p$ *is not locally bounded at the point* $\frac{1}{2}$.

(*As usual* $\psi^{(n)}$ $(n \geqslant 1)$ *are the derivatives of a function* ψ *and* $\psi^{(0)} = \psi$.)

PROOF. (See [1]; the idea about derivatives is due to Professor R. Aron.) Let $\{\theta_n\}$, $\theta_n < \frac{1}{2}$ $(n \in N)$ be a strictly decreasing sequence of positive numbers, converging to 0. Denote $B_n = \{re^{i\theta}: \frac{1}{4} < r < 1, \theta_n < \theta < \theta_{n+1}\}$ $(n \in N)$. It is clear that it is possible to choose the sequence θ_n in such a way that for each $n \in N$ the distance between B_n and the real axis is greater than $\frac{1}{2}^n$. Now choose a sequence of points $\{\zeta_n\}$, $\lim \zeta_n = \frac{1}{2}$, such that $\zeta_n \in B_n$ $(n \in N)$. By the Runge approximation theorem there exists for each $i \in N$ a polynomial ψ_i with the property that

$$\begin{aligned} |\psi_i(\zeta_i)| &\geqslant i, \\ |\psi_i(\zeta)| &\leqslant 1/i! \end{aligned} \qquad (\zeta \in \Delta, \zeta \notin B_i, i \in N).$$

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Let ζ be any point in Δ which does not lie on the real axis. Then there exists a disc, say of radius $\delta > 0$, centered at ζ and contained in Δ, which intersects at most two domains, say B_{i_0}, B_{i_0+1}. So if $n \in N_0$ we have by the Cauchy estimates

$$|\psi_i^{(n)}(\zeta)| \leqslant \frac{n!}{\delta^n} \sup_{|\zeta-\eta|<\delta} |\psi_i(\eta)|$$
$$\leqslant (n!/\delta^n) \cdot (1/i!) \qquad (i \in N, i \neq i_0, i \neq i_0 + 1).$$

Hence for each such ζ the series $\sum |\psi_i^{(n)}(\zeta)|^p$ converges for all $n \in N_0$. Now let ζ lie on the real axis. For each $i \in N$ the disc $\{\eta\colon |\zeta - \eta| < \frac{1}{2}^i\}$ does not intersect B_i by the assumption. By the Cauchy estimates it follows that

$$|\psi_i^{(n)}(\zeta)| \leqslant \frac{n!}{(\frac{1}{2}^i)^n} \sup_{|\zeta-\eta|<\frac{1}{2}^i} |\psi_i(\eta)|$$
$$\leqslant n! \cdot 2^{in} \cdot (1/i!) \qquad (i \in N, n \in N_0)$$

which shows that the series $\sum |\psi_i^{(n)}(\zeta)|^p$ converges for all $n \in N_0$. Hence (i) is satisfied. Further, since $\sum |\psi_i(\zeta_k)|^p \geqslant k^p$ $(k \in N)$ and since $\lim \zeta_k = \frac{1}{2}$ it follows that the sum of the series $\sum |\psi_i(\zeta)|^p$ is not locally bounded at $\frac{1}{2}$, hence (ii) is also satisfied. Q.E.D.

LEMMA 2. *Let $\sum a_i$, $\sum b_i$ be two series with real nonnegative terms. Let $\{p_i\}$, $\{q_i\}$ be two increasing sequences of positive integers. Suppose that $\sum_{j=1}^{p_i} a_j = \sum_{j=1}^{q_i} b_j$ $(i \in N)$. If $\sum a_j$ converges, then $\sum b_j$ also converges and both series have the same sum.*

This is easily established by standard reasoning.

DEFINITION. Let $\{a_i\}$ be a sequence of complex numbers and let $\{p_i\}$ be a sequence of positive integers. Define the sequence $\{b_i\}$ as follows:

$$b_i = \frac{a_n}{p_n} \qquad \left(\sum_{j=1}^{n-1} p_j < i \leqslant \sum_{j=1}^{n} p_j,\ n \in N\right).$$

Then we say that the sequence $\{b_i\}$ is obtained by $\{p_i\}$-splitting of the sequence $\{a_i\}$ and we write $\{b_i\} = \operatorname{spl}(\{a_i\}, \{p_i\})$.

LEMMA 3. *Let $1 \leqslant p < \infty$. There exists a sequence $\{\varphi_i\}$ of complex-valued polynomials such that*

(i) $|\varphi_i^{(n)}(\zeta)| \leqslant 1$ $(i \in N, n \in N_0, \zeta \in \Delta)$;

(ii) $(\varphi_i^{(n)}(\zeta)) \in l^p$ $(n \in N_0, \zeta \in \Delta)$;

(iii) *the sum of the series $\sum |\varphi_i(\zeta)|^p$ is not locally bounded at $\frac{1}{2}$.*

PROOF. Let $\{\psi_i\}$ be the sequence of polynomials given by Lemma 1. Since ψ_i is a polynomial for each $i \in N$ it follows that there exists a sequence of numbers $\{M_i\}$ such that

$$|\psi_i^{(n)}(\zeta)| \leqslant M_i \qquad (i \in N, n \in N_0, \zeta \in \Delta).$$

It is clear that there exists a sequence of numbers $\{N_i\}$ such that (i′) $N_i \geqslant 1$, $N_i \geqslant M_i$ $(i \in N)$,

(ii′) the series $\sum (M_i/N_i)^p$ converges.

For each $i \in N$ let k_i be the largest integer less than or equal to N_i^p. Since $N_i \geqslant 1$ $(i \in N)$ we have $k_i \geqslant 1$ $(i \in N)$. Define

$$\{\varphi_i(\zeta)\} = \mathrm{spl}\,(\{(k_i/N_i)\psi_i(\zeta)\}, \{k_i\}) \qquad (\zeta \in \Delta).$$

It is clear that all φ_i are analytic on Δ and by (i′) we have $|\varphi_i^{(n)}(\zeta)| \leqslant 1$ $(i \in N, n \in N_0, \zeta \in \Delta)$, hence (i) is satisfied. Consider the series $\sum |\varphi_i^{(n)}(\zeta)|^p$. We have

$$\sum_{j=1}^{\sum_{i=1}^m k_i} |\varphi_j^{(n)}(\zeta)|^p = \sum_{i=1}^m \frac{k_i}{N_i^p} |\psi_i^{(n)}(\zeta)|^p \quad (m \in N, n \in N_0, \zeta \in \Delta).$$

Now, $\sum |\psi_i^{(n)}(\zeta)|^p$ converges for each $n \in N_0$, $\zeta \in \Delta$. Since $0 < k_i/N_i^p \leqslant 1$ $(i \in N)$ it follows that the series $\sum (k_i/N_i^p)|\psi_i^{(n)}(\zeta)|^p$ also converges for each $n \in N_0$, $\zeta \in \Delta$. By the last equality Lemma 2 implies that the series $\sum |\varphi_i^{(n)}(\zeta)|^p$ converges for each $n \in N_0$, $\zeta \in \Delta$, hence (ii) is satisfied. By the same equality Lemma 3 also implies that

$$\sum |\varphi_i(\zeta)|^p = \sum \frac{k_i}{N_i^p} |\psi_i(\zeta)|^p \qquad (\zeta \in \Delta).$$

Now we have

$$\left|\sum |\psi_i(\zeta)|^p - \sum |\varphi_i(\zeta)|^p\right| = \sum \left(1 - \frac{k_i}{N_i^p}\right) |\psi_i(\zeta)|^p$$

$$= \sum \frac{N_i^p - k_i}{N_i^p} |\psi_i(\zeta)|^p \leqslant \sum \frac{1}{N_i^p} \cdot M_i^p \qquad (\zeta \in \Delta),$$

where the last series converges by (ii′). Since the sum of $\sum |\psi_i(\zeta)|^p$ is not locally bounded at $\frac{1}{2}$, it follows that also the sum of $\sum |\varphi_i(\zeta)|^p$ is not locally bounded at $\frac{1}{2}$, hence (iii) is also satisfied. Q.E.D.

THEOREM. *Let $1 \leqslant p < \infty$. There exists a function from Δ into l^p with the following properties*:

(i) *it is not continuous*;

(ii) *it is analytic on Δ as a function into $l^{p'}$ for any $p' > p$*;

(iii) *its derivatives (into any $l^{p'}$) of all orders are functions from Δ into l^p.*

PROOF. Let $\{\varphi_i\}$ be the sequence of functions from Lemma 3. Define

$$\{\psi_i(\zeta)\} = \mathrm{spl}\,(\{2^{i(1-1/p)}\varphi_i(\zeta)\}, \{2^i\}) \qquad (i \in N, \zeta \in \Delta).$$

Clearly the functions ψ_i are again analytic on Δ. Let $\alpha > 0$. We have

$$\sum_{j=1}^{\sum_{i=1}^n 2^i} |\psi_j(\zeta)|^{p+\alpha} = \sum_{i=1}^n 2^{-i\alpha/p} |\varphi_i(\zeta)|^{p+\alpha} \qquad (n \in N, \zeta \in \Delta).$$

Since $|\varphi_i(\zeta)| \leqslant 1$ $(\zeta \in \Delta)$ it follows that the convergent series $\sum 2^{-i\alpha/p}$ is majorant for $\sum 2^{-i\alpha/p}|\varphi_i(\zeta)|^{p+\alpha}$ as $\zeta \in \Delta$. Consequently the latter series converges and its sum is bounded on Δ. By Lemma 2 the last equality implies that the series $\sum |\psi_i(\zeta)|^{p+\alpha}$ converges to the same sum. Hence the sum of $\sum |\psi_i(\zeta)|^{p+\alpha}$ is bounded on Δ and by Aron-Cima theorem (see [1]) $\zeta \mapsto (\psi_i(\zeta))$ is an analytic function from Δ into $l^{p'}$ for all $p' > p$. Next we show that $(\psi_i^{(n)}(\zeta)) \in l^p$ $(n \in N_0, \zeta \in \Delta)$. By the definition of ψ_i we have

$$\sum_{j=1}^{\sum_{i=1}^{m} 2^i} |\psi_j^{(n)}(\zeta)|^p = \sum_{j=1}^{m} |\varphi_j^{(n)}(\zeta)|^p \qquad (m \in N, n \in N_0, \zeta \in \Delta)$$

and the conclusion follows by Lemma 3 from the fact that $(\varphi_i^{(n)}(\zeta)) \in l^p$ $(n \in N_0, \zeta \in \Delta)$. Finally by Lemma 2 we also have

$$\sum |\psi_i(\zeta)|^p = \sum |\varphi_i(\zeta)|^p \qquad (\zeta \in \Delta)$$

which shows that the function $\zeta \mapsto (\psi_i(\zeta))$ is not continuous from Δ into l^p since the sum of $\sum |\varphi_i(\zeta)|^p$ is not locally bounded at $\frac{1}{2}$. Hence the function $\zeta \mapsto (\psi_i(\zeta))$ has all required properties. Q.E.D.

REMARK. By observing that $\sum |\psi_i(\zeta)|^{p'}$ above converges uniformly on Δ if $p' > p$ it is easy to see that the Theorem still holds if we replace $l^{p'}$ $(p' \geqslant p)$ by $L^{p'}(S, \Sigma, \mu)$ $(p' \geqslant p)$ provided that Σ contains a disjoint sequence $\{E_n\}$ of sets of measure 1. In this case we define a sequence $\{x_i\}$ of unit vectors in $L^{p'}(S, \Sigma, \mu)$ $(p' \geqslant p)$ by

$$x_i(s) = \begin{cases} 1 & \text{if } s \in E_i, \\ 0 & \text{if } s \notin E_i \end{cases}$$

and define our function as $\zeta \mapsto \sum \psi_i(\zeta) x_i$.

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