A SINGULAR INTEGRAL INEQUALITY ON A BOUNDED INTERVAL

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Abstract. An inequality of the form (1.1) is established, where p, q are real-valued functions on an interval [a, b) of the real line, with \(-\infty < a < b < \infty\), \(p(x) > 0\) on [a, b), \(\mu_0\) is a real number and \(f\) is a complex-valued function in a linear manifold so chosen that all three integrals in (1.1) are absolutely convergent. The problem is singular in that while \(p^{-1} \in L(a, b)\) we require \(q\) to have a behavior at \(b\) of such a form that \(q \notin L(a, b)\).

1. Introduction. An inequality of the form

\[
\int_a^b [p|f'|^2 + q|f|^2] \geq \mu_0 \int_a^b |f|^2 \quad (f \in D)
\]

is established, where \(p, q\) are real-valued functions on an interval [a, b) of the real line, with \(-\infty < a < b < \infty\), \(p(x) > 0\) on [a, b), \(\mu_0\) is a real number and \(f\) is a complex-valued function in a linear manifold so chosen that all three integrals in (1.1) are absolutely convergent. The problem is singular in that while \(p^{-1} \in L(a, b)\) we require \(q\) to have a behavior at \(b\) of such a form that \(q \notin L(a, b)\).

We have established, in a previous paper [1], an inequality of the form (1.1) for the regular case, i.e., \(p^{-1}\) and \(q\) integrable on [a, b], and also for the singular case where \(b = \infty\). Some recent work by Everitt and Giertz [4] and by Kalf [5] make it feasible to study singular inequalities of the form (1.1) on bounded half-open intervals.

The Euler equation for minimizing the left-hand side of (1.1) is

\[
M[y] = \lambda y \quad \text{on} \ [a, b),
\]

where \(\lambda\) is a parameter and \(M[y]\) is the second-order linear differential expression

\[
M[y] = -(py')' + qy \quad \text{on} \ [a, b) \quad (\prime = d/dx).
\]
We make use of certain well-known relationships between equation (1.2) and inequality (1.1) from the calculus of variations, but we do not require that the functions in $D$ for which (1.1) holds satisfy specific boundary conditions at the endpoints $a$ and $b$ as is common when such problems are considered by methods in the calculus of variations.

We use the following notations: $L(a,b)$ and $L^2(a,b)$ denote the classical Lebesgue complex integration spaces; a property is 'loc' on $[a,b)$ if it is satisfied on all compact subintervals of $[a,b)$; AC represents absolute continuity. Thus $AC_{loc}[a,b)$ is the class of all functions that are absolutely continuous on compact subintervals of $[a,b)$.

The coefficient functions $p$ and $q$ are required to satisfy the following basic conditions:

\[
\begin{align*}
\text{(i)} & \quad p \in AC_{loc}[a,b), \quad p(x) > 0, \quad (x \in [a,b)); \\
\text{(ii)} & \quad \text{both } p' \text{ and } q \text{ belong to } L^2_{loc}[a,b). 
\end{align*}
\]

Note that these conditions imply that the differential expression $M$ is regular at all points of $[a,b)$. (See [6, §16.1].) We remark below on the need for (ii).

Following the notation in [1, §2] we define the following sets of functions:

\[
\begin{align*}
\text{(i)} & \quad \Delta = \{ f \in L^2(a,b): f' \in AC_{loc}[a,b), M[f] \in L^2(a,b) \}; \\
\text{(ii)} & \quad \mathcal{D}(\alpha) = \{ f \in \Delta: f(a)\cos \alpha + f'(a)\sin \alpha = 0 \}; \\
\text{(iii)} & \quad D = \{ f \in L^2(a,b): f \in AC_{loc}[a,b), p^{1/2}f', |q|^{1/2}f \in L^2(a,b) \}.
\end{align*}
\]

It should be noted that (ii)(1.3) implies that $\Delta$ contains all infinitely differentiable functions with compact support in $(a,b)$ and so $\Delta$ is dense in $L^2(a,b)$.

For each $\alpha \in [0,\pi)$, an operator $T(\alpha)$ is defined by

\[
\text{domain of } T(\alpha) \text{ is } \mathcal{D}(\alpha) \text{ and } T(\alpha)f = M[f].
\]

It is known that $T(\alpha)$ is selfadjoint in $L^2(a,b)$ if, and only if, $M$ is limit-point at the singular endpoint $b$. (See [6, §18.3].)

Additionally we assume the following conditions on the coefficient functions $p$ and $q$:

\[
\begin{align*}
\text{(i)} & \quad p^{-1} \in L(a,b); \\
\text{(ii)} & \quad \int_a^b q_+ = \infty, \quad \text{where } q_+ = (q + |q|)/2.
\end{align*}
\]

Both conditions are needed in the proof of our theorem. The second condition (ii) insures that $q \not\in L(a,b)$ and so forces $b$ to be a singular endpoint for the differential expression $M$ [6, §16.1]. This then is a distinct departure from the work contained in [1, §2].

Finally, the following conditions are required:
(i) $M$ satisfies the Dirichlet condition at $b$, i.e.,
$$p^{1/2}f' \text{ and } |q|^{1/2}f \in L^2(a,b) \text{ for all } f \in \Delta;$$

(ii) the operator $T(\frac{1}{2}\pi)$ is bounded below in $L^2(a,b)$;

$$\text{i.e., there is a real number } \mu_0 \text{ such that } \langle T(\frac{1}{2}\mu)f,f \rangle \geq \mu_0(\langle f,f \rangle) \text{ for all } f \in \mathcal{D}(\frac{1}{2}\pi),$$

where $\langle , \rangle$ is the usual inner product in $L^2(a,b)$. 

To be exact we define $\mu_0 = \inf \{ \lambda : \lambda \text{ is in the spectrum of } T(\frac{1}{2}\pi) \}$, so that condition (ii) of (1.6) is equivalent to the assumption that $\mu_0 \geq -\infty$.

Specific conditions on the coefficients $p$ and $q$ to insure that $M$ and $T(\frac{1}{2}\pi)$ satisfy (i) and (ii) of (1.6) may be found in the papers of Everitt and Giertz [4] and Kalf [5]. The results of these papers make it reasonable to assume (1.6) as a set of conditions to be satisfied and so indirectly impose conditions on the coefficients $p$ and $q$.

In [4] it is assumed that $p = 1$ on $[a, b)$ and that $q$ satisfies a growth condition near $b$ which insures (ii) of (1.5), (i) and (ii) of (1.6) are satisfied. In [5] a general condition is given that insures that (1.6) is satisfied, but it is then necessary to require $q$ to satisfy (ii) of (1.5).

In [5] Kalf has shown that conditions (ii) of (1.5) and (i) of (1.6) imply that $M$ is strong limit-point at the singular endpoint $b$, i.e.,

$$\lim_{x \to b^-} p(x)f(x)g'(x) = 0 \quad (f, g \in \Delta).$$

An alternative proof of this result may be found in the paper by Everitt [3]. Note that (1.7) implies that $M$ is limit-point at $b$ and so all the operators $T(\alpha) \ (\alpha \in [0, \pi))$ are selfadjoint in $L^2(a,b)$.

We can now state the main result of this paper, which is,

**Theorem 1.** If $p$ and $q$ are real-valued functions for which conditions (1.5) and (1.6) hold, then inequality (1.1) is valid for all functions $f$ in the set $D$ described in (1.4)(iii), with $\mu_0$ the smallest number in the spectrum of the operator $T(\frac{1}{2}\pi)$.

If $\mu_0$ is in the point or point-continuous spectrum of $T(\frac{1}{2}\pi)$, then there is equality in (1.1) if, and only if, $f = c\psi_0$ where $c$ is a complex number and $\psi_0$ is an eigenfunction for $T(\frac{1}{2}\pi)$ corresponding to $\mu_0$.

If $\mu_0$ is in the continuous spectrum of $T(\frac{1}{2}\pi)$, then there is equality in (1.1) if, and only if, $f$ is the zero function. The inequality is the best possible in the sense that there is a sequence $\{f_n\}$ such that $f_n \in D$, $\int_a^b |f_n|^2 = 1 \ (n = 1, 2, \ldots)$ and

$$\lim_{n \to \infty} \int_a^b [p|f_n'|^2 + q|f_n|^2] = \mu_0.$$ 

Our proof of Theorem 1 depends upon the following approximation theorem.

**Theorem 2.** If the hypothesis of Theorem 1 holds, then for each $\epsilon > 0$ and each real-valued function $f$ in $D$ there is a real-valued function $g$ in $\mathcal{D}(\frac{1}{2}\pi)$ such that
\[ \left| \int_a^b pf'^2 - \int_a^b pg'^2 \right| < \varepsilon, \quad \left| \int_a^b f^2 - \int_a^b g^2 \right| < \varepsilon, \quad \left| \int_a^b qf^2 - \int_a^b qg^2 \right| < \varepsilon. \]

The proofs of these theorems are in the next section.

2. Proofs. We restrict our attention in this section to real-valued functions since it is sufficient to prove (1.1) for all real-valued functions in the set \( D \); this is a consequence of taking the coefficients \( p \) and \( q \) to be real-valued on \([a, b)\).

We use the following lemma in our proof of Theorem 2; see also [1, §3].

**Lemma.** If the hypothesis of Theorem 1 holds, \( f \in D \) and \( \varepsilon > 0 \), then there is a number \( X \) in \((a, b)\) with the property that

\[ (2.1) \quad \int_a^b pf'^2 < \varepsilon, \quad \int_a^b |q|^2 < \varepsilon, \quad \int_a^b f^2 < \varepsilon, \]

and

\[ (2.2) \quad f^2(X) \int_a^X |q| < \varepsilon, \quad (X - a)f^2(X) < \varepsilon. \]

**Proof.** The fact that (2.1) holds follows from the fact that \( f \in D \). To obtain (2.2) we first observe that \( f \) in \( D \) and (1.6)(i) imply that \( \lim_{x \to b^-} f(x) \) exists. Indeed,

\[ f(x) = f(a) + \int_a^x f' = f(a) + \int_a^x p^{-1/2} p^{1/2} f'; \]

the last integral converges as \( x \to b^- \) since \( p \to x^{1/2} \) and \( p^{1/2} f' \in L^2(a, b) \).

If \( \lim_{x \to b^-} f(x) \neq 0 \), then there exist numbers \( k > 0 \) and \( t \in (a, b) \) such that \( f^2(x) > k \) for \( x \in (t, b) \). Then

\[ \int_a^X q^+ f^2 \geq \int_a^t q^+ f^2 + k \int_t^X q^+. \]

This inequality and (1.5)(ii) imply that \( \int_a^b q^+ f^2 = \infty \) which is contrary to the assumption that \( |q|^2 f \in L^2 \); i.e., that \( f \in D \). Therefore \( \lim_{x \to b^-} f(x) = 0 \).

It now follows that \( (x - a)f^2(x) \to 0 \) as \( x \to b^- \). To complete the proof of the lemma it is sufficient to show that there is a sequence \( \{x_n\} \), \( x_n \to b^- \), for which \( f^2(x_n) \int_a^{x_n} |q| \to 0 \) as \( n \to \infty \). If no such sequence exists, then there is a \( d > 0 \) and \( t \in (a, b) \) such that

\[ f^2(x) > d \int_a^X |q| \quad (x \in (t, b)). \]

Multiplying by \( |q| \) and integrating from \( t \), we obtain

\[ \int_t^X |q| f^2 \geq d \left[ \log \int_a^X |q| - \log \int_t^X |q| \right]. \]

This inequality, (1.5)(ii) and (1.6)(i) are incompatible with \( f \in D \).

**Proof of Theorem 2.** For a positive number \( \varepsilon_1 \) we choose \( X \) in \((a, b)\) so that the conclusion of the lemma is valid. Then we note that \( f \in D \) implies that.
G L^2(a,X) and that the set of continuously differentiable functions vanishing together with their derivatives at a and X is dense in $L^2(a,X)$. Therefore, for $\eta > 0$, we may choose a continuously differentiable function $\phi$ such that $\phi(a) = \phi'(a) = \phi(X) = \phi'(X) = 0$, $\phi(x) = 0$ on $[X,b)$ and $\int_a^X |f' - \phi| < \eta$.

The function $g$ is defined by

$$g(x) = -\int_x^X \phi \quad (x \in [a,X]),$$

$$g(x) = 0 \quad (x \in (X,b)).$$

It is clear that $g \in \mathcal{D}(\frac{1}{2})$ since $g'(a) = 0$, $g$ has a continuous second derivative, and $p$, $p'$ and $q$ are all in $L^2(a,X)$, in view of the conditions (1.3). We note here that conditions (1.3) are essential to our argument, as were similar conditions in [1, §3], and that it remains undecided whether condition (ii) of (1.3) could be replaced by the weaker assumption that $p'$ and $q$ are locally integrable.

Using the function $g$ defined above and the lemma of this section, the estimates obtained in [1, §3] remain valid, with $\infty$ replaced by $b$, and the proof of Theorem 2 proceeds in the same way as the proof of Theorem 3 of [1].

**Proof of Theorem 1.** The proof of Theorem 1 is accomplished in three stages. The first is to establish inequality (1.1) for functions $f$ in $\mathcal{D}(\frac{1}{2})$, the second is to extend the validity of the inequality to all of $D$, and the third is to determine the cases of equality. Each of these steps is similar to a corresponding part of the proof of Theorem 2 of [1] and we therefore limit our discussion here to a statement of the basic ideas involved and points where something needs to be added to the previous argument.

We begin by showing that (1.1) holds for $f$ in $\mathcal{D}(\frac{1}{2})$. Indeed, for such an $f$ an integration by parts, an application of (1.7) and condition (1.6) yield

$$\int_a^b [pf'^2 + qf^2] = \int_a^b fM[f] \geq \mu_0 \int_a^b f^2,$$

which establishes (1.1) for $f$ in $\mathcal{D}(\frac{1}{2})$. Since this argument appears to be identical to the one used in [1] it should be pointed out that an analogue of (1.7) is used in [1] and the arguments used in proving this analogue cannot be used in proving (1.7).

To prove that (1.1) holds for functions in $D$, we assume there is a function in $D$ for which (1.1) fails to hold and use the results of Theorem 2 to obtain a contradiction of the fact that (1.1) holds for all functions in $\mathcal{D}(\frac{1}{2})$. The details are the same as those used in [1].

The third stage of the proof, that of determining the cases of equality, separates into two cases according as $\mu_0$ is an eigenvalue or not.

If $\mu_0$ is in the point spectrum or point-continuous spectrum, then it is clear that there is equality in (1.1) for any constant multiple of an eigenfunction corresponding to $\mu_0$. Conversely, if equality holds in (1.1) for some $f$ in $D$, then it follows from Theorem 4 of [1] that $f$ is a solution of the differential equation $M[f] = \mu_0 f$, and since the differential expression $M$ is in the limit point case at $b$ it follows that $f = c \psi_0$, where $\psi_0$ is an eigenfunction corresponding to $\mu_0$ and $c$ is a constant.
If \( \mu_0 \) is in the continuous spectrum, we assume there is a function \( f \) in \( D \), \( f \neq 0 \) for which equality holds. Again using Theorem 4 of [1] we find that \( f \) is a solution of equation (1.2) for \( \lambda = \mu_0 \) and apply (1.7), Dirichlet condition (1.6)(i) and an integration by parts to conclude that \( f(a) f'(a) = 0 \). Since \( \mu_0 \) is not an eigenvalue for \( T(\frac{1}{2}\pi) \), \( f'(a) \neq 0 \); therefore, \( f(a) = 0 \) and \( \mu_0 \) is an eigenvalue of the operator \( T(0) \). That this is impossible is established as in [1] using analytic properties of the Weyl \( m \)-coefficients \( m(\lambda, 0) \) and \( m(\lambda, \frac{1}{2}\pi) \) presented by Chaudhuri and Everitt in [2] since those properties hold equally well for a differential expression \( M[f] \) considered on a bounded interval \( [a, b] \) with a singular endpoint \( b \).

3. Examples. We conclude with an example. Consider inequality (1.1) with

\[
(3.1) \quad a = 0, \quad b = 1, \quad p(x) = 1, \quad q(x) = \frac{3}{4}(1 - x)^2 \quad (x \in [0, 1]).
\]

Then the conditions imposed by Everitt and Giertz [4] are satisfied as is the Dirichlet condition (1.6)(i) at the singular point 1. Therefore, the differential expression \( M[f] \) for this example is strong limit-point at 1. Also, \( \int_0^1 q \, dx = \int_0^1 q = \infty \). Thus all the conditions are satisfied and (1.1) holds for the example described by (3.1).

We write the differential equation (1.2) as

\[
(3.2) \quad -y'' + (\nu^2 - \frac{1}{4})(1 - x)^{-2}y = \lambda y \quad (0 < x < 1),
\]

where \( \nu = 1 \). Then the Weyl \( m \)-coefficient \( m(\lambda, \frac{1}{2}\pi) \) is given by

\[
m(\lambda, \frac{1}{2}\pi) = 2J_1(s)/(J_1(s) + 2sJ_1'(s)),
\]

where \( s^2 = \lambda, \quad 0 \leq \arg \lambda < 2\pi, \quad 0 \leq \arg s < \pi, \) and \( J_1 \) is the Bessel function of order 1 of the first kind. (See Titchmarsh [7, §4.8].) We note that \( m \) is meromorphic and therefore the spectrum is discrete. Moreover, \( \mu_0^{1/2} \) is the first positive zero of \( J_1(s) + 2sJ_1'(s) \). The equalizing function (eigenfunction) is then given by

\[
\psi(\gamma) = (1 - x)^{1/2}J_1(s(1 - x)) \quad (0 \leq x < 1).
\]

References


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