SCALAR CURVATURES ON $O(M), G_2(M)$

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Abstract. We show that every $C^\infty f : G_2(M) \to \mathbb{R}, M^n$ a compact connected riemannian manifold $n \geq 3$, is the scalar curvature function of some complete riemannian metric on $G_2(M)$, the grassmann bundle of 2 planes over $M$, except possibly when $K = \text{constant} \geq 0$. A similar result holds for $O(M)$ bundle of orthonormal frames on $M$.

This note is an application of Theorem A of [3] and O'Neill's formula for the curvature of a riemannian submersion [5], [4] and [1]. Theorem C of [3] gives an affirmative answer to the question described in the abstract if $f(P) < 0$ for at least one 2-plane section $P$ tangent to $M$.

Preliminaries. Let $(M^n, ds^2)$ be a compact riemannian manifold and let $O(M)$ be the principal $O(n)$ bundle of $ds^2$-orthonormal frames on $M$. Choose a connection of it: $OM \to M$.

We assume the setting of [1, Section 1]. (See also [4].) Briefly, let $\langle , \rangle$ be the bi-invariant metric on $O(n)$ defined via the positive definite $-B$ (killing form) on the lie algebra $\mathfrak{g}$ of $O(n)$, and $\gamma$ the connection form for $\pi$. Split each $X \in T(OM)$ into horizontal and vertical components relative to $\gamma$: $X = X^H + X^K$. For $t > 0$, $g_t = \pi^* ds^2 + t^2 \langle \gamma, \gamma \rangle$ is a family of $O(n)$-right invariant complete riemannian metrics on $P$ and relative to each one $\pi$ is a riemannian submersion on $(M, ds^2)$. Fix a $t > 0$ and let $g \equiv g_t$. Let $U$ be open in $M$ and $X_1, \ldots, X_n$ be a $ds^2$-orthonormal (o.n. from now on) frame on $U$. Consider reductive decomposition of $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$, orthogonal relative to $-B$ where $\mathfrak{h}$ is the lie algebra of $O(2) \times O(n-2)$, and $N$ is the space of skew symmetric matrices of the form:

\[
\begin{pmatrix}
0 & 0 & -\xi^3 \\
0 & 0 & -\eta^r \\
\xi & \eta & 0
\end{pmatrix}, \quad \xi, \eta \text{ column vectors in } R^{n-2},
\]

(see [2, vol. II, p. 280]).

Let $e_1^r, e_2^s, r, s = 1, \ldots, n-2$, be the obvious $(-B)$-o.n. basis of $\mathfrak{n}$, where

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\( e_i' \) is the matrix we obtain above if \( \xi \) is the \( r \text{th} \) standard basis element of \( R^{n-2} \) and \( \eta = 0 \). Similarly with \( e_2' \). Let \( \tilde{H} = \tilde{H}_I + \tilde{H}_{n-2} \) with \( \tilde{H}_I = \{(r, 0'), r \in R\} \) the lie algebra of \( O(2) \) and \( \tilde{H}_{n-2} \) the lie algebra of \( O(n-2) \).

Let \( X'_{n+1}, \ldots, X'_{3n-4}; X'_{3n-3}; X'_{3n-2}, \ldots, X'_{n+r} \) be notation for an o.n. basis of \( N + \tilde{H}_I + \tilde{H}_2 \), where \( r = \text{dim } O(n) \). If \( A \rightarrow A^* \) denotes the lie algebra monomorphism \( \hat{G} \rightarrow \mathfrak{o}(OM) \), the vector fields of \( O(M) \), determined by the free \( O(n) \) action, we have that \( X_1, \ldots, X_n; r^{-1}X'_{n+1}, \ldots, r^{-1}X'_{n+r} \) form a local g-o.n. basis on \( \pi^{-1}(U) \), where \( X_i \) is the \( \gamma \)-horizontal lift of \( X'_i \).

Consider now the following

\[
\begin{align*}
&O(M), g \xrightarrow{\pi} \frac{O(M)}{O(2) \times O(n-2)} = G_2(M), \hat{g} \\
&\quad M, ds^2 = M, ds^2
\end{align*}
\]

where \( \hat{g} \) is the metric on \( G_2(M) \) with respect to which \( p \) is a riemannian submersion. Let \( u \in \pi^{-1}(U) \subset O(M), p(u) \in G_2(M), x = \pi(u) \in M \). We want to calculate \( S_g(p(u)) \): the scalar curvature at \( p(u) \) relative to \( \hat{g} \).

A \( \hat{g} \)-o.n. frame at \( p(u) \) is obtained by projecting a g-o.n., \( g \)-horizontal (i.e., normal to the \( p \)-fibre relative to \( g \)) frame at \( u \) of \( O(M) \). From \( qp = \pi \) follows that \( \text{Horiz} (p) = \text{Horiz} (\pi) + \text{span } \{X_{n+1}, \ldots, X_{3n-4} \} \) \( X_{n+s} = r^{-1}X'_{n+s} \) (sum orthogonal relg). From now on let \( 1 \leq a, \beta \leq n + r, 3n - 3 \leq a \leq n + r, n + 1 \leq b, \eta, \theta \leq n + r, n + 1 \leq \lambda, \mu \leq 3n - 4, 1 \leq i, j, k \leq n \).

Let \( \bar{K}_{\alpha \beta} = K_{\hat{g}}(p_*(X_\alpha), p_*(X_\beta)) \) where \( X_\alpha \) is one of the \( X_i \)'s or \( r^{-1}X'_i \)'s and \( \bar{K}_{\hat{g}} \) is the sectional curvature relative to \( \hat{g} \). By O'Neill's formula for the curvature of a riemannian submersion ([5], [4], [1]),

\[
(1) \quad \bar{K}_{\alpha \beta} = K_{\hat{g}}(X_\alpha, X_\beta) + \frac{3}{2} \|[[X_\alpha, X_\beta]]^V\|^2,
\]

where \( V \) stands for "vertical part" or "\( g \)-orthogonal projection onto the \( p \)-fibre" in any \( T_u(OM) \). Notice that

\[
[[X_i, X_j]]^V = \sum_a g([[X_i, X_j], X_a])X_a = \sum_a i^2 \langle \gamma[X_i, X_j], \gamma(X_a) \rangle X_a
\]

\[
= \sum_a \langle \gamma[X_i, X_j], X'_a \rangle X'_a.
\]

If \( \| \| \| \) is the length relative to \( g \),

\[
\|[[X_i, X_j]]^V\|^2 = i^2 \sum_a \langle \gamma_i^{a}, X'_a \rangle^2,
\]

where \( \gamma_i^{a} = \gamma[X_i, X_j] \in \hat{G} \).

Let \( \langle \gamma_{ij}, X'_a \rangle(u) = 2H^a_{ij}(u) \) and obtain

\[
(2) \quad \bar{K}_j = K_j + 3i^2 \sum_a (H^a_{ij})^2.
\]

Similarly, \( \bar{K} \equiv K_{\hat{g}}(p_*(X_i), p_*(X_i)) = K(X_i, X_i) + \frac{3}{2} \|[[X_i, X_i]]^V\|^2 \). Here \( X_i \) is \( \pi- \)
horizontal and \( X_\lambda \) is a fundamental \( p \)-vertical. \( \vdots \) \([X_i, X_\lambda] \) is zero.

\( (3) \quad K_{i\lambda} = K(X_i, X_\lambda). \)

Now, \( K_{\lambda, \mu} = K(X_\lambda, X_\mu) + \frac{1}{4} ||[X_\lambda, X_\mu]_v||^2 \). Recall that \( X_\lambda = t^{-1} X_\lambda^* \) and the basis \( X'_\lambda \) was exactly the \((\epsilon_1', \epsilon_2')\)s and \((\epsilon_3', \epsilon_4')\)s above. It is \([\epsilon_1', \epsilon_2'] = [\epsilon_3', \epsilon_4'] = 0 \) and

\[
[\epsilon_1', \epsilon_2'] = \delta_{rs} \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{in} \quad \mathcal{H}_1 \subset \mathcal{G}.
\]

If \(-B\) is so normalized as to have \( \epsilon_1', \epsilon_2 \) of length one then \([\epsilon_1', \epsilon_2']\) is also of length 1.

\([X_\lambda, X_\mu] = [t^{-1} X_\lambda^*, t^{-1} X_\mu^*] = t^{-2}[X_\lambda^*, X_\mu^*].\]

From the above discussion \( ||[X_\lambda, X_\mu]_v||^2 = (t^{-2})^2 t^2 1 \) or it is zero depending on \( \lambda, \mu \). The number of nonzero ones is \( n - 2 \).

From this and (2), (3) we have:

\[
\sum_{\alpha, \beta} K_{\alpha, \beta} = \sum_{i < j} K_{ij} + \sum_{\lambda < \mu} K_{\lambda, \mu} + \sum_{\lambda < \mu} K_{\lambda, \mu}
\]

\[
(4) = \sum_{i < j} \left\{ K_{ij} + 3t^2 \sum_a (H_{ij}^a)^2 \right\} + \sum_{i, \lambda} K_{i, \lambda}
\]

\[
+ \sum_{\lambda < \mu} K_{\lambda, \mu} + \frac{3}{4} t^{-2} (n - 2).
\]

Now from O'Neill's theorem [5] on the submersion \( \pi \) one can express \( K_{ij}, K_{i, \lambda}, K_{\lambda, \mu} \) relative to the curvature of \( M \) and the \( \pi \)-fibre \( O(n) \).

\( K_{ij} = K(X_i, X_j) = K_{d_\pi^2}(X_i', X_j') - \frac{3}{4} ||[X_i, X_j]^W||^2 \)

where \( W \) denotes the vertical component relative to the \( \pi \)-fibre:

\[
[X_i, X_j]^W = \sum_b g([X_i, X_j], X_b) X_b
\]

\[
= \sum_b \langle [X_i, X_j], X_b^* \rangle X_b^* = 2 \sum_b H_{ij}^b X_b^*,
\]

\[
||[X_i, X_j]^W||^2 = 4t^2 \sum_b (H_{ij}^b)^2, \quad (H_{ij}^b = H_{ij}^b(u)).
\]

\( (5) \quad K_{ij} = K_{d_\pi^2}(X_i', X_j') - 3t^2 \sum_b (H_{ij}^b)^2. \)

\( K_{i, \lambda} = t^2 \| (\nabla_{X_i^*} X_\lambda^*) \|_v^2 \| ([5], [4]) \) where \( \nabla \) is the riemannian connection corresponding to \( g \), and hor stands for \( \pi \)-horizontal component. It is easy to see from [5, Lemma 2, p. 446] or [4], that
\[\|(\nabla_{\lambda} X_{\lambda})_{\text{hor}}\|^2 = \sum_j \frac{1}{2} g([X_i, X_j], X_\lambda)^2 \]

(6)

\[K_{i\lambda} = \frac{1}{2} \sum_j (2 H_{ij}^\lambda)^2,\]

Note. This is the same result as in Proposition 5 of [1] where \(K_{i\lambda} = R_{i\lambda i\lambda}\). \(K_{\lambda\mu} = K(X_\lambda, X_\mu)\) and it must be the curvature of the same section considered as tangent to the totally geodesic \(\pi\)-fibre, i.e.,

\[K_{\lambda\mu} = \frac{1}{4} t^{-2} \|[X^\lambda, X^\mu]\|^2\]

where \(\|\|\) is the original \(O(n)\) norm. (This agrees with Proposition 5 of [1]:

\[K_{\lambda\mu} = R_{\lambda\mu\lambda\mu} = \frac{1}{4} t^{-2} \sum_f (C_{\lambda\mu f}^\lambda)^2,\]

where \(C_{st} = \langle[X_s, X_t], X_r\rangle = C_{rt}^s\), etc.) But \(\|[X^\lambda, X^\mu]\| = 1\) or 0 as above and there are exactly \((n - 2)\) combinations \((\lambda < \mu)\) that give us 1:

(7) \[\sum_{\lambda < \mu} K_{\lambda\mu} = \frac{1}{4} t^{-2}(n - 2).\]

From (4), (5), (6), (7) we have:

\[\sum_{\alpha, \beta} \bar{R}_{\alpha\beta} = \sum_{i < j} K_{\alpha\beta\alpha\beta}(X_i, X_j) - \sum_{i < j} 3t^2 \sum_b (H_{ij}^b)^2 + \sum_{i < j} 3t^2 \sum_a (H_{ij}^a)^2 + \sum_{i < j} t^2 \sum_j (H_{ij}^\lambda)^2 + \frac{1}{4} t^{-2}(n - 2) + \frac{3}{4} t^{-2}(n - 2).\]

Therefore,

\[S_\sigma(p(u)) = S_{\sigma^2}(\pi(u)) - 2t^2 \sum_{\lambda} \sum_{i < j} (H_{ij}^\lambda(u))^2 + t^{-2}(n - 2)\]

after collecting terms and observing the ranges of the indices \(a, b,\) and \(\lambda\).

\((S_{\sigma^2}(\pi(u))\) is the scalar curvature of \(d\sigma^2\) on \(M\) at \(\pi(u)\).)

\[\Lambda(U) = 2 \sum_{\lambda} \sum_{i < j} (H_{ij}^\lambda(u))^2 \geq 0 \text{ for all } u \in \pi^{-1}(U).\]

Let \(S\) be the notation for the scalar curvature of \(d\sigma^2\) on \(M\), \(S: M \to R\) and \(\Lambda: O(M) \to R\) but it factors through \(M\) by the \(O(n)\)-invariance of \(g\). So, instead of \(\Lambda(u)\) we write \(\Lambda(x), x = \pi(u)\).

We proved:

**Proposition.** The scalar curvature of \(g\) at \(p(u)\) is equal to \(S(x) - t^2 \Lambda(x) + t^{-2}(n - 2)\) where \(x = \pi(u), \) and \(\Lambda(x) \geq 0.\)
Theorem. A $C^\infty$ function $f: G_2(M) \to \mathbb{R}$ is a scalar curvature function for some Riemannian metric on $G_2(M)$ except perhaps when it is a nonnegative constant.

The proof of this theorem is an application of

**Theorem A ([3]).** Let $(N, g)$ be a smooth compact Riemannian manifold with Gaussian (resp. scalar if $\dim N \geq 3$) curvature $S$ and let $f \in C^\infty(N)$. If there is a constant $c > 0$ such that

$$\min cf < S(x) < \max cf$$

for all $x \in N$, then there is a smooth metric $g_1$ on $N$ with Gaussian (resp. scalar) curvature $f$.

Recall that

$$2H_{ij}^\lambda = \langle \gamma[X_i, X_j], X_\lambda \rangle$$

and $X_i, X_j$ are $\gamma$-horizontal vectors in $T(OM)$ of unit length relative to $g = g_t$ and therefore of unit length relative to every $g_t, t > 0$. The length of $\gamma[X_i, X_j]$ in $G$ is bounded independent of $t$, i.e.,

$$0 \leq \Lambda_m \leq \Lambda(x) \leq \Lambda_M$$

for all $x \in M$ with $\Lambda_m, \Lambda_M$ constants independent of $t$.

Since $S(x)$ is bounded, it follows that $\tilde{S}_m(t)/\tilde{S}_M(t) \to 1$ as $t \to 0$, where $\tilde{S}_m(t)$ and $\tilde{S}_M(t)$ are the maximum and minimum of the scalar curvature of $g_t$, and the proof is complete.

**Special case.** If $f = \text{constant} > 0$ is the sectional curvature of $(M, ds^2)$ then

$$M = S^n/\Gamma, \quad \Gamma \subset O(n + 1) \quad \text{and} \quad O(M) = O(n + 1)/\Gamma,$$

where $\Gamma$ is a finite subgroup of $O(n + 1)$ [6, p. 69]. If $M$ is 1-connected, $M = S^n$ of radius $f^{-1/2}$ and therefore $G_2(M) = O(n + 1)/O(2) \times O(n - 2)$, which admits a constant scalar curvature function as a homogeneous space of $O(n + 1)$. Any $C^\infty$ function from $G_2(M) \to \mathbb{R}$ will then be a scalar curvature for some Riemannian metric on $G_2(M)$, by Theorem C of [3].

The following is proved exactly the same way as the above theorem.

**Proposition.** A $C^\infty f: O(M) \to \mathbb{R}$ is a scalar curvature function for some Riemannian metric on $O(M)$, except perhaps when $f = \text{constant} \geq 0$.

**Proof.** Using the same notation conventions,

$$K_{ij} = K_{ds^2}(X'_i, X'_j) - 3t^2 \sum_\eta (H^\eta_{ij})^2,$$

$$K_{i\theta} = t^2 \sum_j (H^\theta_{ij})^2,$$

$$K_{\eta\theta} = \frac{1}{4}t^{-2}\langle[X_\eta^*, X_\theta^*], [X_\eta^*, X_\theta^*]\rangle = t^{-2}K_0(X_\eta^*, X_\theta^*)$$

where $K_0$ is the sectional curvature of the fibre $O(n)$ in its original metric $\langle , \rangle$.

Therefore,

$$S_g(u) = S(x) - t^2 \Lambda_1(u) + t^{-2}c$$

where $S_g(u)$ is the scalar curvature of $O(M)$ relative to the metric $g = g_t$ at $u$.

$$\Lambda_1(u) = 2 \sum_{i<j} \sum_\eta (H^\eta_{ij}(u))^2 \geq 0$$
bounded independent of \( t \) and \( c \) is the constant positive scalar curvature of \((O(m), \langle \cdot, \cdot \rangle)\).

If \( 0 \leq f_m < f_M \) by the exact same procedure as in the theorem we obtain that \( f \) is a scalar curvature on \( O(M) \).

In the particular case that \( f \) is \( K \circ p \) with \( K \) the constant positive curvature of \( M \), then \( M = S^n/T \) and \( O(M) = O(n + 1)/T \) is a homogeneous space of a compact Lie group that admits a metric with positive constant scalar curvature. By Theorem C of [3], all \( C^\infty \) functions on \( O(M) \) are scalar curvatures and in particular \( f \).

References


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