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HYPERSPACES OF TOPOLOGICAL VECTOR SPACES: THEIR EMBEDDING IN TOPOLOGICAL VECTOR SPACES

PREM PRAKASH AND MURAT R. SERTEL

ABSTRACT. Let L be a real (Hausdorff) topological vector space. The space $\mathcal{K}[L]$ of nonempty compact subsets of L forms a (Hausdorff) topological semivector space with singleton origin when $\mathcal{K}[L]$ is given the uniform (equivalently, the finite) hyperspace topology determined by L. Then $\mathcal{K}[L]$ is locally compact iff L is so. Furthermore, $\mathcal{K}\mathcal{Q}[L]$, the set of nonempty compact convex subsets of L, is the largest pointwise convex subset of $\mathcal{K}[L]$ and is a cancellative topological semivector space. For any nonempty compact and convex set $X \subset L$, the collection $\mathcal{K}\mathcal{Q}[X] \subset \mathcal{K}\mathcal{Q}[L]$ is nonempty compact and convex. L is iseomorphically embeddable in $\mathcal{K}\mathcal{Q}[L]$ and, in turn, there is a smallest vector space $\mathcal{L}$ in which $\mathcal{K}\mathcal{Q}[L]$ is algebraically embeddable (as a cone). Furthermore, when L is locally convex, $\mathcal{L}$ can be given a locally convex vector topology $\mathcal{T}$ such that the algebraic embedding of $\mathcal{K}\mathcal{Q}[L]$ in $\mathcal{L}$ is an iseomorphism, and then $\mathcal{L}$ is normable iff L is so; indeed, $\mathcal{T}$ can be so chosen that, when L is normed, the embedding of L in $\mathcal{K}\mathcal{Q}[L]$ and that of $\mathcal{K}\mathcal{Q}[L]$ in $\mathcal{L}$ are both iseometries.

1. **Preliminaries.** R denotes the set of real numbers with the usual topology, and $R_+ = \{\lambda \in R | \lambda \geqslant 0\}$. For any set X, $[X]$ denotes the set of nonempty subsets of X. When X is a topological space, $\mathcal{K}[X]$ denotes the set of compact nonempty subsets of X. When X lies in a real vector space, $\mathcal{Q}[X]$ denotes the set of convex nonempty subsets of X. Finally, when X lies in a real topological vector space, $\mathcal{K}\mathcal{Q}[X] = \mathcal{K}[X] \cap \mathcal{Q}[X]$.

In topologizing hyperspaces (i.e., spaces of subsets), we will use the uniform topology, regarding which we adopt Michael [1] as standard reference. Let X be a uniform space, and let $\{E_\alpha \subset X \times X | \alpha \in \mathcal{C}\}$ be a fundamental system of symmetric entourages of X. The *uniform topology* for $[X]$ is the topology generated by declaring $\mathcal{E}_\alpha[A] = \{B \in [X] | B \subset E_\alpha(A) \text{ and } A \subset E_\alpha(B)\}$ for each $\alpha \in \mathcal{C}$ to be a nbd of A $(A \in [X])$. By the uniform topology on a hyperspace $\mathcal{K}[X] \subset [X]$ is meant the relative topology of $\mathcal{K}[X]$ when $[X]$ carries the uniform topology.

1.0 DEFINITION [2]. Let $(S, \oplus)$ be a commutative semigroup and $\Psi: R_+ \times S \to S$ a map such that, denoting $\Psi(\lambda, s) = \lambda s$,

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$$\lambda(\mu s) = (\lambda \cdot \mu)s \qquad \text{(left action)},$$

$$1s = s \qquad \text{(unitariness)},$$

$$\lambda(s \oplus t) = \lambda s \oplus \lambda t \qquad \text{(homomorphism)}$$

for all λ, $\mu \in R_+$ and $s, t \in S$. We call S a *semivector space*. When S is a Hausdorff space and the operations $\oplus$ and Ψ are both continuous, we call S a *topological* semivector space.

Thus, real vector spaces are all semivector spaces, so that the topological vector spaces we speak of are those with Hausdorff topology.

2. **Semivector hyperspaces of topological vector spaces.** Let L be a real vector space, and e its identity element. Now $[L]$ is a semivector space with identity $\{e\}$ when $A \oplus B = \{a + b | a \in A, b \in B\}$ and $\lambda A = \{\lambda a | a \in A\}$, where $+$ stands for vector addition in L $(A, B \in [L], \lambda \in R_+)$. Furthermore, $\mathcal{Q}[L] \subset [L]$ is also a semivector space and is *pointwise convex*, i.e., $\{A\}$ is convex for each $A \in \mathcal{Q}[L]$. In fact $\mathcal{Q}[L]$ is the largest pointwise convex subset of $[L]$: If $A \in [L]$ and $\lambda A \oplus \lambda' A \subset A$ for each $\lambda = (1 - \lambda') \in [0, 1]$, then $A \subset L$ must be convex.

From here on, L will always be a topological vector space.

Now $\mathcal{K}[L] \subset [L]$ is a semivector subspace and $\mathcal{KQ}[L]$ is the largest pointwise convex semivector subspace of $\mathcal{K}[L]$. Also, the *origin* $0[L] = 0\mathcal{K}[L] = 0\mathcal{Q}[L] = 0\mathcal{KQ}[L] = \{\{e\}\}$ is singleton. *N. B.* The uniform topology on $\mathcal{K}[L]$ coincides with the finite topology [1, p. 153, 1.1 and p. 160, 3.3].

2.1. PROPOSITION. (1) $\mathcal{K}[L]$ *is a topological semivector space, locally compact iff L is.* (2) *The map* ℓ: $x \mapsto \{x\}$ $(x \in L)$ *iseomorphically embeds L into the topological semivector subspace* $\mathcal{KQ}[L] \subset \mathcal{K}[L]$.

PROOF. (ad (1)). $\mathcal{K}[L]$ is Hausdorff as L is (see [1, p. 164, 4.9.8]) and will be locally compact iff L is locally compact (see [1, p. 164, 4.9.12]). This leaves to show only the continuity of the operations $\oplus$ and Ψ of $\mathcal{K}[L]$. The continuity of vector addition $+$: $L \times L \to L$ implies the continuity of the map $\hat{+}$: $[L \times L] \to [L]$ defined by $\hat{+}(P) = \{a + b | (a, b) \in P\}$ $(P \in [L \times L])$ (see [1, p. 169, 5.9.1]). Thus, the restriction of $\hat{+}$ to the space $\mathcal{B} = \{C \times D | C, D \in \mathcal{K}[L]\} \subset \mathcal{K}[L \times L]$ of compact boxes is also continuous. Furthermore, the Cartesian product $\pi(C, D) = C \times D$ is continuous on $\mathcal{K}[L] \times \mathcal{K}[L] \to \mathcal{B}$ (see Theorem 3 of [3]). Now $\oplus$ is simply the composition $\oplus = \hat{+} \circ \pi$: $\mathcal{K}[L] \times \mathcal{K}[L] \to \mathcal{K}[L]$, and so is continuous. Similarly, the continuity of scalar multiplication $R_+ \times L \to L$ implies that of scalar multiplication Ψ: $R_+ \times \mathcal{K}[L] \to \mathcal{K}[L]$.

(ad (2)). From (1) it follows that the space $\mathcal{KQ}[L] \subset \mathcal{K}[L]$ is a topological semivector space. Now the map ℓ is a homeomorphism [1, p. 155, 2] and is easily checked to be a homomorphism. □

2.2 PROPOSITION. (1) $\mathcal{KQ}[L]$ *is cancellative.* (2) *In fact, for all A, B, C* $\in [L]$, *whenever* (i) *B is bounded and C is closed and convex, or* (ii) *B is compact and C is open and convex, we have* $B \subset B \oplus C \Rightarrow e \in C$ (*or, equivalently,* $B \oplus A \subset B \oplus C \Rightarrow A \subset C$).

PROOF. (ad (1)). Since $\mathcal{KQ}[L]$ is a pointwise convex (Hausdorff) topological semivector space with singleton origin, (1) follows from [2, Theorem 2.11]; it also follows directly from (2)(i), proved below.

(ad (2)). Let B, $C \in [L]$ with C convex.

(i) Assume B bounded and C closed, and that $B \subset B \oplus C$. Now, for any $b(i) \in B$, we have $b(i) = b(i+1) + c(i+1)$ for some $b(i+1) \in B$ and $c(i+1) \in C$ $(i = 0, 1, 2, \dots)$, so that $b(0) - b(n) = \sum_N c(i)$ $(N = \{1, \dots, n\}; n = 1, 2, \dots)$ and, denoting $\bar{c}(n) = n^{-1} \sum_N c(i)$, the convexity of C gives $\bar{c}(n) \in C$, while the boundedness of B implies that $\bar{c}(n) = n^{-1}(b(0) - b(n))$ converges to e as n goes to ∞. As C is closed, the limit point $e \in C$.

(ii) Assume B compact and C open, and that $B \subset B \oplus C$. Denoting $C_b = \{b\} \oplus C$ for every $b \in B$, $\{C_b | b \in B\}$ is then an open cover of B, admitting a finite subcover $\{C_{b(i)} | i \in M\}$. Now, for every $i \in M$, there is a $j \in M$, such that $b(i) \in \{b(j)\} \oplus C$ and, hence

$$\sum_N b(i) \in \boldsymbol{\Sigma}_N (\{b(j)\} \oplus C)$$

for some subset $N \subset M$, where $\boldsymbol{\Sigma}$ is to $\sum$ as $\oplus$ is to $+$. As C is convex, we may write

$$\boldsymbol{\Sigma}_N (\{b(j)\} \oplus C) = \Big(\sum_N b(j)\Big) \oplus nC$$

(see [2, p. 122]). Denoting $\bar{b} = n^{-1} \sum_N b(j)$, where $n = \#N$, we therefore have $\bar{b} \in \{\bar{b}\} \oplus C$, whereby $e \in C$.

(To see the claimed equivalence, first note that, if $B \oplus A \subset B \oplus C \Rightarrow A \subset C$ for all $A \in [L]$, then the case of $A = \{e\}$ yields $B \subset B \oplus C \Rightarrow e \in C$. For the rest, assume $B \subset B \oplus C \Rightarrow e \in C$ whenever (i) or (ii), and take any $A \in [L]$ with $B \oplus A \subset B \oplus C$. Pick any $a \in A$, so that $B \oplus \{a\} \subset B \oplus C$, hence $B \subset B \oplus C \oplus \{a^-\}$, where a^- is the additive inverse of a. Now $C \oplus \{a^-\}$ is closed and convex (resp., open and convex) whenever C is so. Thus, whether (i) or (ii) is the case, $e \in C \oplus \{a^-\}$, i.e., $a \in C$, showing $A \subset C$.) □

2.3 THEOREM. *If $X \subset L$ is nonempty compact and convex, then $\mathcal{KQ}[X] \subset \mathcal{KQ}[L]$ is (nonempty) compact and convex.*

PROOF. Let $X \subset L$ be nonempty compact and convex. The uniform topology which the (uniform space) X determines for $\mathcal{K}[X]$ yields $\mathcal{K}[X]$ compact Hausdorff, since X is compact Hausdorff (see [1, p. 160, 3.3 and p. 164, 4.9.12]). Furthermore, $\mathcal{K}[X]$ inherits the same topology as a subspace of $\mathcal{K}[L]$ as it receives from X (see [1, p. 167, 5.2.3 and 5.2.3′]), so that $\mathcal{K}[X] \subset \mathcal{K}[L]$ is compact Hausdorff.

Now $\mathcal{KQ}[X] \subset \mathcal{K}[X]$ is clearly nonempty and convex, since X is so. This leaves only to show that $\mathcal{KQ}[X] \subset \mathcal{K}[X]$ is closed. To that end, let $\mathfrak{F}$ be a converging filterbase in $\mathcal{KQ}[X]$. Since $\mathcal{K}[X]$ is compact Hausdorff, the limit point, say Q, is unique and $Q \in \mathcal{K}[X]$. We show that Q is also convex.

For each $\lambda \in [0, 1]$, denote $\lambda' = (1 - \lambda)$ and define the map Ω_λ on $\mathcal{K}[X]$ through $\Omega_\lambda(P) = \lambda P \oplus \lambda' P$ $(P \in \mathcal{K}[X])$. Since scalar multiplication in L is continuous, for each $\lambda \in [0, 1]$, $\lambda\mathcal{K}[X] \subset \mathcal{K}[L]$ and Ω_λ is a continuous mapping of $\mathcal{K}[X]$ into $\mathcal{K}[L]$; in fact, from the convexity of X one easily sees that $\Omega_\lambda(\mathcal{K}[X]) \subset \mathcal{K}[X]$ and that the restriction of Ω_λ to $\mathcal{KQ}[X]$ is nothing but the identity map of $\mathcal{KQ}[X]$. Also, given a $P \in \mathcal{K}[X]$, if $\Omega_\lambda(P) \subset P$ for each $\lambda \in [0, 1]$, then $P \in \mathcal{KQ}[X]$. Take any $\lambda \in [0, 1]$. We show that $\Omega_\lambda(Q) = Q$. Let $\mathcal{V} \subset \mathcal{K}[X]$ be any nbd of $\Omega_\lambda(Q) \in \mathcal{K}[X]$. As Ω_λ is continuous, there is a nbd $\mathcal{U} \subset \mathcal{K}[X]$ of $Q \in \mathcal{K}[X]$ such that $\Omega_\lambda(\mathcal{U}) \subset \mathcal{V}$. As $\mathfrak{F}$ converges to Q, there is some $\mathcal{W} \in \mathfrak{F}$ with $\mathcal{W} \subset \mathcal{U}$. But $\mathcal{W} \subset \mathcal{KQ}[X]$, so that $\mathcal{W} = \Omega_\lambda(\mathcal{W}) \subset \Omega_\lambda(\mathcal{U}) \subset \mathcal{V}$. This shows that $\mathfrak{F}$ converges to $\Omega_\lambda(Q)$; and, the limit point being unique, $\Omega_\lambda(Q) = Q$. Then $Q \in \mathcal{KQ}[X]$, showing that $\mathcal{KQ}[X]$ is closed and completing the proof. □

3. **Embedding $\mathcal{KQ}[L]$ in a topological vector space.** A subset of the semivector space $[L]$, to be embeddable in a vector space, must clearly be pointwise convex and cancellative. Now the largest pointwise convex set in $[L]$ is $\mathcal{Q}[L]$, but clearly $\mathcal{Q}[L]$ fails to be cancellative and is, therefore, not embeddable in a vector space. On the other hand, we have just extended the operations of L to $\mathcal{KQ}[L]$ (see 2.1), and this is a topological semivector space which is both pointwise convex and cancellative (2.2). In standard fashion (see also [2, 2.9]) we embed it in

The real vector space $\mathcal{L}$: Denoting $\mathcal{S} = \mathcal{KQ}[L] \times \mathcal{KQ}[L]$, equip $\mathcal{S}$ with coordinatewise addition $(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$ and define the equivalence relation $\mathcal{G} \subset \mathcal{S}$ through $(A, B)\mathcal{G}(C, D) \Leftrightarrow A \oplus D = B \oplus C$, so that $\mathcal{G}$ is a semigroup congruence and the quotient $\mathcal{L} = \mathcal{S}/\mathcal{G}$ is a group. Denote the equivalence class of (A, B) by $[A, B]$, and define scalar multiplication ψ: $R \times \mathcal{L} \to \mathcal{L}$ by setting $\psi(\lambda, [A, B]) = [\lambda A, \lambda B]$ if $\lambda \geqslant 0$ and $\psi(\lambda, [A, B]) = [|\lambda| B, |\lambda| A]$ if $\lambda \leqslant 0$. Now $\mathcal{L}$ is a real vector space and the map $\mathfrak{g}$ which sends each $A \in \mathcal{KQ}[L]$ to the equivalence class $[2A, A] \in \mathcal{L}$ is an algebraic isomorphism embedding $\mathcal{KQ}[L]$ into $\mathcal{L}$. Evidently, $\mathcal{L}$ is, up to an isomorphism, the smallest vector space in which $\mathcal{KQ}[L]$ may be algebraically embedded. *N. B.* Clearly, $[A, A] = [B, B]$ for all A, $B \in \mathcal{KQ}[L]$, and this equivalence class is the identity element of $\mathcal{L}$.

From here on L will always be locally convex.

We now take a fundamental system $\mathcal{U} = \{U_\alpha | \alpha \in \mathcal{Q}\}$ of symmetric open convex nbds of the identity e in L, and for $\mathcal{L}$ we define

The topology $\mathfrak{T}$: For each $\alpha \in \mathcal{Q}$, declare $\mathcal{W}_\alpha = \{[A, B] \in \mathcal{L} | B \subset A \oplus U_\alpha, A \subset B \oplus U_\alpha\}$ to be an open nbd of the identity element $[A, A]$ of $\mathcal{L}$; and, for each $[P, Q] \in \mathcal{L}$, declare $[P, Q] \oplus \mathcal{W}_\alpha$ to be an open nbd of $[P, Q]$. (We check that, given $[A, B] \in \mathcal{W}_\alpha$ and $(C, D) \in [A, B]$, $D \subset C \oplus U_\alpha$ and $C \subset D \oplus U_\alpha$: As $(C, D) \in [A, B]$, we have $A \oplus D = B \oplus C$, while $[A, B] \in \mathcal{W}_\alpha$ implies $A \subset B \oplus U_\alpha$, whereby $A \oplus D \subset B \oplus D \oplus U_\alpha$, so that $B \oplus C \subset B \oplus D \oplus U_\alpha$, from which 2.2(2)(ii) implies $C \subset D \oplus U_\alpha$; similarly, $D \subset C \oplus U_\alpha$.)

3.1. THEOREM. (1) $\mathcal{L}$ *equipped with the topology* $\mathfrak{T}$ *is a topological vector space, and* (2) $\mathfrak{g}$ *embeds* $\mathcal{KQ}[L]$ *iseomorphically in* $\mathcal{L}$.

PROOF. (ad (1)). To see that the family $\mathcal{W} = \{\mathcal{W}_\alpha | \alpha \in \mathcal{Q}\}$ is a local base for a Hausdorff vector topology on $\mathcal{L}$, we note that each $\mathcal{W}_\alpha$ is symmetric, and check that:

(i) For each pair $\alpha, \beta \in \mathcal{Q}$, there is a $\gamma \in \mathcal{Q}$ such that $\mathcal{W}_\gamma \subset \mathcal{W}_\alpha \cap \mathcal{W}_\beta$: Choose $\gamma \in \mathcal{Q}$ such that $U_\gamma \subset U_\alpha \cap U_\beta$.

(ii) For each $\alpha \in \mathcal{Q}$, there is a $\beta \in \mathcal{Q}$ such that $\mathcal{W}_\beta \oplus \mathcal{W}_\beta \subset \mathcal{W}_\alpha$: Choose $\beta \in \mathcal{Q}$ such that $U_\beta \oplus U_\beta \subset U_\alpha$.

(iii) For each $\alpha \in \mathcal{Q}$, there is a $\beta \in \mathcal{Q}$ such that $\lambda\mathcal{W}_\beta \subset \mathcal{W}_\alpha$ for each scalar $\lambda \in R$ with $|\lambda| \leqslant 1$: Choose $\beta \in \mathcal{Q}$ such that $\lambda U_\beta \subset U_\alpha$ for each $\lambda \in R$ with $|\lambda| \leqslant 1$.

(iv) Given any $[A,B] \in \mathfrak{L}$ and $\alpha \in \mathcal{Q}$, there is a $\lambda \in R$ such that $[A,B] \in \lambda\mathcal{W}_\alpha$: Taking any $b \in B$, for each $a \in A$ find $\lambda_a \in R$ such that $a \in \lambda_a U_\alpha \oplus \{b\}$, where we may assume $\lambda_\alpha > 0$ since U_α is symmetric. Then, for each $a \in A$, $a \in \lambda_a U_\alpha \oplus B$, and so $\{\lambda_a U_\alpha \oplus B | a \in A\}$ is an open cover of A and, since $A \subset L$ is compact, there is a finite subcover $\{\lambda_{a(i)} U_\alpha \oplus B | i = 1, \ldots, m\}$. Defining $\lambda_A = \mathrm{Max}\{\lambda_{a(1)}, \ldots, \lambda_{a(m)}\}$, now $A \subset \lambda_A U_\alpha \oplus B$. Finding λ_B in similar fashion and setting $\lambda = \mathrm{Max}\{\lambda_A, \lambda_B\}$ we see that $[A,B] \in \lambda\mathcal{W}_\alpha$.

(v) $\bigcap_{\mathcal{Q}}\mathcal{W}_\alpha = \{[A,A]\}$ (where $[A,A]$ is the identity element of $\mathfrak{L}$): $[A,A] \in \bigcap_{\mathcal{Q}}\mathcal{W}_\alpha$, since $[A,A] \in \mathcal{W}_\alpha$ for each $\alpha \in \mathcal{Q}$. On the other hand, if B, $C \in \mathcal{K}\mathcal{Q}[L]$ are distinct, then there is a $\beta \in \mathcal{Q}$ such that $B \not\subset C \oplus U_\beta$ or $C \not\subset B \oplus U_\beta$, so that $[B,C] \notin \mathcal{W}_\beta$ and $[B,C] \notin \bigcap_{\mathcal{Q}}\mathcal{W}_\alpha$.

(ad (2)). Having already seen that $\mathfrak{g}$ is an algebraic isomorphism, all we need to check here is that $\mathfrak{g}$ is continuous and open. A basic open nbd of an element $P \in \mathcal{K}\mathcal{Q}[L]$ is of the form $\mathcal{U}_\alpha(P) = \{Q \in \mathcal{K}\mathcal{Q}[L] | P \subset Q \oplus U_\alpha, Q \subset P \oplus U_\alpha\}$ $(\alpha \in \mathcal{Q})$. A basic open nbd of $\mathfrak{g}(P) = [2P,P] \in \mathfrak{L}$ according to the subspace topology of $\mathfrak{g}(\mathcal{K}\mathcal{Q}[L])$ determined by $\mathfrak{T}$ is of the form $\mathcal{W}'_\alpha(P) = ([2P,P] \oplus \mathcal{W}_\alpha) \cap \mathfrak{g}(\mathcal{K}\mathcal{Q}[L])$ $(\alpha \in \mathcal{Q})$. What we actually show now is the formula $\mathfrak{g}(\mathcal{U}_\alpha(P)) = \mathcal{W}'_\alpha(P)$.

Let $[2Q,Q] \in \mathfrak{g}(\mathcal{U}_\alpha(P))$, so that $P \subset Q \oplus U_\alpha$ and $Q \subset P \oplus U_\alpha$. Let

$$[A,B] = [2Q,Q] \oplus [P,2P] = [2Q \oplus P, Q \oplus 2P],$$

so that $A \oplus Q \oplus 2P = B \oplus 2Q \oplus P$, i.e., $A \oplus P = B \oplus Q$. Now $A \oplus P \subset A \oplus Q \oplus U_\alpha$, so we have $B \oplus Q \subset A \oplus Q \oplus U_\alpha$, and 2.2(2)(ii) then yields $B \subset A \oplus U_\alpha$. Similarly, $A \subset B \oplus U_\alpha$, so that $[A,B] \in \mathcal{W}_\alpha$ and $[2Q,Q] = [2P,P] \oplus [A,B] \in \mathcal{W}'_\alpha(P)$, i.e., $\mathfrak{g}(\mathcal{U}_\alpha(P)) \subset \mathcal{W}'_\alpha(P)$. Now take any element of $\mathcal{W}'_\alpha(P)$, i.e., a point

$$[2Q,Q] = [2P,P] \oplus [A,B] = [2P \oplus A, P \oplus B]$$

with $A \subset B \oplus U_\alpha$ and $B \subset A \oplus U_\alpha$. Then $2Q \oplus P \oplus B = Q \oplus 2P \oplus A$, so that 2.2(2)(ii) gives $Q \oplus B = P \oplus A \subset P \oplus B \oplus U_\alpha$ and $Q \subset P \oplus U_\alpha$. Similarly, $P \subset Q \oplus U_\alpha$, so $Q \in \mathcal{U}_\alpha(P)$ and $[2Q,Q] = \mathfrak{g}(Q) \in \mathfrak{g}(\mathcal{U}_\alpha(P))$, showing $\mathcal{W}'_\alpha(P) \subset \mathfrak{g}(\mathcal{U}_\alpha(P))$. We conclude that $\mathfrak{g}(\mathcal{U}_\alpha(P)) = \mathcal{W}'_\alpha(P)$, and this completes the proof. □

3.2 THEOREM. *$\mathfrak{L}$ with the topology $\mathfrak{T}$ is locally convex.*

PROOF. W.l.o.g., we may assume that, for each $\alpha \in \mathcal{Q}$, U_α is convex, circled, and radial at e and that, for each nonzero $\lambda \in R$, $\lambda U_\alpha \in \mathcal{U}$. Let $\alpha \in \mathcal{Q}$. It is straightforward to check that (i) $\mathcal{W}_\alpha$ is circled and (ii) for each nonzero $\lambda \in R$, $\lambda\mathcal{W}_\alpha \in \mathcal{W}$. To check that (iii) $\mathcal{W}_\alpha$ is convex, let $[A,B]$, $[C,D] \in \mathcal{W}_\alpha$ and $\lambda = (1 - \lambda') \in [0,1]$. Now $\lambda[A,B] \oplus \lambda'[C,D] = [\lambda A \oplus \lambda' C, \lambda B \oplus \lambda' D]$; and, since U_α is convex, we have $\lambda U_\alpha \oplus \lambda' U_\alpha = U_\alpha$. Now $[A,B]$, $[C,D] \in \mathcal{W}_\alpha$ says $A \subset B \oplus U_\alpha$ and $C \subset D \oplus U_\alpha$, so that

$$\lambda A \oplus \lambda' C \subset \lambda B \oplus \lambda' D \oplus \lambda U_\alpha \oplus \lambda' U_\alpha .$$

Similarly, $\lambda B \oplus \lambda' D \subset \lambda A \oplus \lambda' C \oplus U_\alpha$. Thus, $[\lambda A \oplus \lambda' C, \lambda B \oplus \lambda' D] \in \mathcal{W}_\alpha$, showing that $\mathcal{W}_\alpha$ is convex. This in conjunction with (iv) in the proof of 3.1(1) implies that (iv) $\mathcal{W}_\alpha$ is radial at the identity element $[A, A]$ of $\mathfrak{L}$. Thus, $\mathcal{W}$ is a local base for a (unique) locally convex topology in $\mathfrak{L}$. □

3.3 Theorem (Rådström [4]). (1) *$\mathfrak{L}$ with the topology $\mathfrak{T}$ is normable iff L is normable, and* (2) *if L is normed, $\mathfrak{L}$ admits a norm for which $\mathfrak{f}$ and $\mathfrak{g}$ are isometries.*

Proof. (ad (1)). "Only if" is obvious from the conjunction of 2.1(2) and 3.1(2). To see "if", assume that L is normed by a norm ρ, so that $V = \{x \in L | \rho(x) < 1\} = U_\alpha$ for some $\alpha \in \mathfrak{A}$. Thus,

$$\mathcal{W}_\alpha = \{[A, B] \in \mathfrak{L} | A \subset B \oplus V, B \subset A \oplus V\} \in \mathcal{W}.$$

Since V is radial at the origin, circled, convex and bounded, one easily checks (see also the proof of 3.1(1)) that $\mathcal{W}_\alpha$ has these properties too, so that (the Hausdorff space) $\mathfrak{L}$ is normable, proving (1).

(ad (2)). In fact, the Minkowski functional ρ^* of $\mathcal{W}_\alpha$ is a norm for $\mathfrak{L}$ and, computing that $\rho^*[2P, P] = \mathrm{Sup}_P \rho(p)$ for each $P \in \mathcal{KQ}[L]$, one easily sees $\mathfrak{f}$ and $\mathfrak{g}$ to be isometries. □

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References

1. Ernest Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182. MR **13**, 54.

2. Prem Prakash and Murat R. Sertel, *Topological semivector spaces: Convexity and fixed point theory*, Semigroup Forum **9** (1974), 117–138. MR **51**, 1556.

3. ———, *On the continuity of Cartesian product and factorisation*, Discussion Paper no. 82, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Ill., 1974. (Also issued as Preprint Series No. I/ 74–16, Easter 1974, International Institute of Management, D–1000 Berlin 33, Griegstrasse 5.)

4. Hans Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169. MR **13**, 659.

Graduate School of Management, Northwestern University, Evanston, Illinois 60201

International Institute of Management, 1 Berlin 33, Griegstrasse 5–7, Germany