PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

REPRESENTING HOMOLOGY CLASSES OF CLOSED ORIENTABLE SURFACES

MARK D. MEYERSON

ABSTRACT. We prove that the primitive elements of the first homology group of a genus g torus can be represented by a simple closed curve.

Let T be a closed orientable surface with g handles ($g \geqslant 0$). The first homology group, $H_1(T)$, is the free abelian group on $2g$ generators. In general, if an element of $H_1(T)$ is represented by a closed curve, the curve must cross itself. However, we prove that if the element is primitive, then it can be represented by a simple closed curve. Primitive means not divisible by an integer greater than one. Our proof is accomplished through use of the "twist" homeomorphism described by Lickorish [1].

Let α_k and β_k be the standard oriented simple closed curves around the kth handle of T for $1 \leqslant k \leqslant g$. So each α_i or β_j represents a basis element of $H_1(T)$. For $1 \leqslant k < g$, let γ_k be a simple closed curve looping the kth and $(k+1)$st handles of T. The Lickorish "twist" homeomorphism about a simple closed curve J is defined by cutting T along J, giving a full twist to one edge, and then gluing T back together along J. Twists about α_k, β_k, and γ_k induce automorphisms on $H_1(T)$ carrying $(\cdots, a_k, b_k, a_{k+1}, b_{k+1}, \ldots)$ to $(\cdots, a_k \pm b_k, b_k, a_{k+1}, b_{k+1}, \ldots)$, $(\cdots, a_k, b_k \pm a_k, a_{k+1}, b_{k+1}, \ldots)$, and $(\cdots, a_k \pm b_k \pm b_{k+1}, b_k, a_{k+1} \pm b_k \pm b_{k+1}, b_{k+1}, \ldots)$ respectively. The "±" signs appear since there is always a choice of two ways of twisting.

Now let $(a_1, b_1, a_2, b_2, \ldots)$ be a fixed element of $H_1(T)$. By several appropriate iterations of twists about α_k's and β_k's, and using the Euclidean algorithm, we get a homeomorphism which induces a map carrying $(a_1, b_1, a_2, b_2, \ldots)$ to $(d_1, 0, d_2, 0, \ldots)$, where $d_i = \text{g.c.d.}\{a_i, b_i\}$. By following this by further twists about α_k's, β_k's, and γ_k's, we can produce a self-homeomorphism of T which induces a map carrying $(a_1, b_1, a_2, b_2, \ldots)$ to $(d, 0, 0, 0, \ldots)$, where

$$d = \text{g.c.d.}\{d_1, d_2, \ldots\} = \text{g.c.d.}\{a_1, b_1, a_2, b_2, \ldots\}.$$

For $d = \pm 1$, $(d, 0, 0, 0, \ldots)$ is represented by the simple closed curve $\pm\alpha_1$, and by applying the inverse of the homeomorphism we have constructed to α_1 we get the simple closed curve we desire.

The converse is true, namely simple closed curves represent primitive (or trivial) elements. The following argument is due to Hans Samelson. A

Received by the editors March 25, 1975.
AMS (MOS) subject classifications (1970). Primary 57A05.

nonbounding simple closed curve J (i.e., not representing a trivial element) is always intersected by some other closed curve exactly once. Just take a small arc crossing J once and connect the end points in the arcwise connected set $T-J$. But if J represents a homology class divisible by m, then its intersection number with any other closed curve is a multiple of m. So J represents a primitive element.

References

1. W.B.R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) **76** (1962), 531–540. MR **27** #1929.

Department of Mathematics, Stanford University, Stanford, California 94305

Current Address: Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801