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CENTRAL SEQUENCES IN FLOWS ON 2-MANIFOLDS OF FINITE GENUS

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ABSTRACT. Let ϕ be a continuous flow on the metric space X and let X^1, $X^2, \ldots$ denote the "central" sequence of closed ϕ-invariant subsets of X obtained by iterating the process of taking nonwandering points of ϕ. A. Schwartz and E. Thomas have proved that, if X is an orientable 2-manifold of finite genus, then this sequence can have not more than two distinct elements. We extend this result to include the nonorientable case; then this sequence can have at most three distinct elements. Analogous results are derived for the sequences obtained by iterating the processes of taking α and ω limit sets, or closures of α and ω limit sets.

1. **Introduction.** In [6] Schwartz and Thomas determine bounds on the lengths of the various "central sequences" in an arbitrary continuous flow on an orientable 2-manifold of finite genus. The purpose of this note is to extend these results to include the nonorientable case.

Before stating these results precisely we need several definitions. Let X be a metric space, and let ϕ: $X \times \mathbf{R}^1 \to X$ be a continuous flow on X. For $t \in \mathbf{R}^1$ let ϕ_t denote the homeomorphism of X defined by $\phi_t(x) = \phi(x, t)$ $(x \in X)$. A point $x \in X$ is *nonwandering* if $x \in J^+(x)$; here $J^+(x)$ denotes the set of all limits of sequences $\{\phi(x_n, t_n)\}$, where $\{x_n\}$ converges to x and $\{t_n\}$ tends to infinity. If $Y \subseteq X$ is ϕ-invariant (i.e., $\phi_t(Y) = Y$ for all $t \in \mathbf{R}^1$) then $y \in Y$ is *nonwandering relative to* Y if y is a nonwandering point of the flow $(Y, \phi|_Y)$; we let Y' denote the set of points that are nonwandering relative to Y.

DEFINITION 1 (BIRKHOFF [1]). Define recursively: $X^0 = X$; $X^{\alpha+1} = (X^\alpha)'$ if α is an ordinal and X^α is defined; and $X^\alpha = \bigcap_{\beta<\alpha} X^\beta$ if α is a limit ordinal and X^β is defined for all $\beta < \alpha$. The transfinite sequence $\alpha \to X^\alpha$ of nested, closed, ϕ-invariant subsets of X is called the 1-*central sequence* of ϕ; the least ordinal α such that $X^{\alpha+1} = X^\alpha$ is called the *depth* of the 1-central sequence, and is denoted $d_1(\phi)$.

A point $y \in X$ is called an ω-*limit point* (α-*limit point*) of $x \in X$ if there is a sequence of real numbers $t_k \to \infty$ $(t_k \to -\infty)$ such that $\phi_{t_k}(x) \to y$. The set of ω-limit points (α-limit points) of x is denoted $\Omega(x)$ $(A(x))$. A point $x \in X$ is said to be P^+ *stable* (P^- *stable*) if $x \in \Omega(x)$ $(x \in A(x))$; x is *Poisson stable* if $x \in \Omega(x) \cap A(x)$. If $Y \subseteq X$ is ϕ-invariant let Y^* denote the set of points of Y which are either α- or ω-limit points of some point of Y.

DEFINITION 2 (MAIER [3]). Define recursively:

$$X^{[0]} = X; \quad X^{[\alpha+1]} = \mathrm{cl}\big((X^{[\alpha]})^*\big)$$

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(here cl denotes closure) if α is an ordinal and $X^{[\alpha]}$ is defined; $X^{[\alpha]} = \bigcap_{\beta<\alpha} X^{[\beta]}$ if α is a limit ordinal and $X^{[\beta]}$ is defined for all $\beta < \alpha$. The sequence $\alpha \to X^{[\alpha]}$ is called the 2-*central sequence* of ϕ; its *depth*, denoted $d_2(\phi)$, is the least ordinal α such that $X^{[\alpha+1]} = X^{[\alpha]}$.

DEFINITION 3 (BIRKHOFF-SMITH [2]). Define recursively: $X^{(0)} = X$; $X^{(\alpha+1)} = (X^{(\alpha)})^*$ if α is an ordinal and $X^{(\alpha)}$ is defined; $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ if α is a limit ordinal and $X^{(\beta)}$ is defined for all $\beta < \alpha$. The sequence $\alpha \to X^{(\alpha)}$ is called the 3-*central sequence* of ϕ; its *depth*, denoted $d_3(\phi)$, is the least ordinal α such that $X^{(\alpha+1)} = X^{(\alpha)}$.

We are interested in the case in which X is a 2-manifold (separable metric, connected, and possibly with nonempty boundary) of finite genus. A 2-manifold X is said to have *finite genus* if there is a compact 2-manifold $Y \subseteq X$ such that $X - Y$ is homeomorphic to a subset of $\mathbf{R}^2$; in this case the *genus* of X is defined to be the genus of such a submanifold Y (cf. [5]). A 2-manifold is *closed* if it is compact and without boundary.

With this terminology the results of [6] can be stated as follows:

THEOREM (SCHWARTZ-THOMAS). *If X is an orientable 2-manifold of finite genus and ϕ is a continuous flow on X, then $d_1(\phi)$, $d_2(\phi)$, $d_3(\phi) \leqslant 2$.*

If nonorientable manifolds are admitted, the corresponding result is:

THEOREM. *If X is a 2-manifold of finite genus and ϕ is a continuous flow on X, then $d_1(\phi) \leqslant 3$ and $d_2(\phi)$, $d_3(\phi) \leqslant 2$.*

An example is given in [6] in which $d_1 = 3$ and $d_2 = d_3 = 2$, so these bounds cannot be improved. The restriction to finite genus is necessary: in [4] it is proved that if α, β, $\gamma \geqslant 2$ are any countable ordinals with $\alpha \geqslant \beta$, then there is an orientable open 2-manifold X and a C^∞ flow ϕ on X, with $d_1(\phi) = \alpha$, $d_2(\phi) = \beta$, and $d_3(\phi) = \gamma$. Also, if X is *any* manifold of dimension greater than two and α, β, γ are as above, then there is a flow ϕ on X with $d_1(\phi) = \alpha$, $d_2(\phi) = \beta$, and $d_3(\phi) = \gamma$ [4].

2. **Preliminaries**. We first indicate that the "topological preliminaries" of [6] hold as well in the nonorientable case. The following proposition is an easy consequence of Theorem 3 of [5].

PROPOSITION 1. *Suppose that X is a 2-manifold without boundary and that* genus$(X) = n$. *Then X is homeomorphic to the surface obtained by removing a closed, totally disconnected set from a closed 2-manifold of genus n.*

We note that a closed 2-manifold X of genus n (orientable or not) has the property that any collection of $n + 1$ disjoint simple closed curves in X separates X. Using this fact and Proposition 1, we may prove, just as in [6]:

PROPOSITION 2 (SCHWARTZ-THOMAS). *Suppose that X is a closed 2-manifold. Let $\Gamma \subseteq X$ be a continuum (i.e., compact, connected subset) which meets a closed arc T in a totally disconnected set. Let W be a component of $X - \Gamma$ and let $\mathfrak{T}$ be the collection of components of $T - \Gamma$ lying in W. If $\mathfrak{T}$ is infinite, say $\mathfrak{T} = \{T_i | i \in \mathbf{Z}\}$, then, for almost all i, there is an open cell $W_i \subseteq W$ such that the boundary of W_i is contained in $T_i \cup \Gamma$, and $T_i \subseteq$ cl$(W - W_i)$.*

We list two further facts which will also be used in the proof of the main theorem. In each case the proof is straightforward and hence omitted.

Suppose that X is a 2-manifold and let ϕ be a continuous flow on X. Let T be a closed arc in X such that, for some $\varepsilon > 0$, ϕ is a homeomorphism of $T \times [-\varepsilon, \varepsilon]$ onto its image in X. Such an arc is called a *transversal* to ϕ. We define the *first return map* f on T as follows: $x \in \operatorname{domain}(f)$ if $\phi_t(x) \in T$ for some $t > 0$; in this case let $t_x = \min\{t > 0 | \phi_t(x) \in T\}$ and set $f(x) = \phi_{t_x}(x)$.

PROPOSITION 3. *Suppose ϕ is a continuous flow on the 2-manifold X and that T is a transversal to ϕ with first return map f. Suppose that f is monotone increasing on* $\operatorname{domain}(f)$ *and that* $y \in \operatorname{domain}(f)$. *Then $y \in X^1$ only if the orbit of y is periodic.*

Now suppose y_0 and y_1 are points of domain(f), and set $y_0' = f(y_0)$, $y_1' = f(y_1)$. For $i = 0, 1$, let A_i denote the subarc of the orbit $O(y_i)$ through y_i from y_i to y_i'. Let B (B') denote the subarc of T from y_0 to y_1 (y_0' to y_1'). We say that A_0 and A_1 are *parallel* if there is an open disc $W \subseteq X$ such that the boundary of W is $A_0 \cup A_1 \cup B \cup B'$.

PROPOSITION 4. *Suppose that ϕ is a continuous flow on the closed 2-manifold X and that T is a transversal to ϕ with first return map f. Let $\{y_i\}$ be a sequence of points of* domain(f); *for $i \in \mathbf{Z}^+$ let $y_i' = f(y_i)$ and let A_i denote the subarc of $O(y_i)$ from y_i to y_i'. Suppose that both $\{y_i\}$ and $\{y_i'\}$ converge monotonically. Then, for almost all i, A_i and A_{i+1} are parallel.*

3. **Proof of the main theorem**. We first extend Theorem 6.2 of [6] to the nonorientable case.

PROPOSITION 5. *Suppose X is a 2-manifold of finite genus and ϕ is a continuous flow on X. If $x \in X^1$, but x is not P^+ or P^- stable, then $A(x)$ and $\Omega(x)$ contain no regular points.*

PROOF. We follow the proof of 6.2 in [6] until the last step. We may assume that X is closed, that $x \in X^1$ is not P^+ or P^- stable, but that there is a regular point y in $\Omega(x)$. Let Γ denote the continuum $\Omega(x)$, and let T be a transversal to ϕ, with y an interior point of T. By assumption $x \notin \Gamma$. If $\Gamma \cap T$ is infinite, then the component W of $X - \Gamma$ that contains $O(x)$ meets T in an infinite collection of open arcs. By Proposition 2, $O(x)$ crosses one of these into an open, positively (or negatively) invariant 2-cell, and this contradicts $x \in X^1$. Thus we may assume that in fact $\Gamma \cap T = \{x\}$.

Hence we may choose points $x_i \in O(x) \cap T$ so that both $\{x_i\}$ and $\{x_i' = f(x_i)\}$ converge monotonically to y (f denotes the first return map on T). For $i \in \mathbf{Z}^+$, let A_i denote the subarc of $O(x)$ from x_i to x_i'. By Proposition 4 we may assume that all the A_i are parallel.

In the case that both $\{x_i\}$ and $\{x_i'\}$ decrease (say) monotonically to y, we see that the first return map on the subinterval $[y, x_1]$ of T is monotone increasing. As $x_i \in X^1$, this leads to a contradiction of Proposition 3.

In case $\{x_i\}$ decreases to y and $\{x_i'\}$ increases to y we may argue as follows. Pick a sequence of points $z_i \in O(x) \cap T$ so that $\{z_i\}$ increases to y and $\{z_i' = f(z_i)\}$ decreases to y. For $i \in \mathbf{Z}^+$ let B_i denote the subarc of $O(x)$ from z_i to z_i'; we may assume all the B_i are parallel. It follows that, for some i, the first return map on the subinterval $[y, x_i]$ of T is monotone increasing, and we have a contradiction as before.

PROOF OF THE MAIN THEOREM. The conclusions $d_2(\phi) \leqslant 2$ and $d_3(\phi) \leqslant 2$ follow easily from Proposition 5. We prove $d_1(\phi) \leqslant 3$.

If this fails for any 2-manifold of finite genus, then we can construct, as in [6], a flow on a closed 2-manifold for which it fails. Hence we assume that X is closed and that there is a point $x \in X^3 - X^4$. Let T be a transversal to ϕ, with x an interior point of T, and let f denote the first return map on T. Since $x \notin X^4$ we may assume that no P^+ or P^- stable orbit meets T. By Proposition 5, we may assume that if $y \in X^1 \cap T$ then $O(y) \cap T$ is finite.

Pick a sequence $\{y_i\} \subseteq T$ of points on distinct orbits of $X^2 - X^3$, so that $\{y_i\}$ converges monotonically to x and so that the set $\{\bigcup_i O(y_i) \cap T\}$ has x as an accumulation point. Let Γ denote the continuum $\limsup_{i\to\infty} O(y_i)$. Exactly as in [6] we can show that the y_i may be chosen so that no $O(y_i)$ lies in Γ, and hence so that $\Gamma \cap T$ is finite. Thus, by choosing T sufficiently small, we may assume $\Gamma \cap T = \{x\}$. In this case the only accumulation point of the sequence $\{y_i' = f(y_i)\}$ is x, so we may assume that $\{y_i\}$ and $\{y_i'\}$ both converge monotonically to x.

If both sequences decrease to x we obtain a contradiction just as in the proof of Proposition 5. So we assume that $\{y_i\}$ decreases to x and $\{y_i'\}$ increases to x. Let $\{T_i\}$ be a sequence of disjoint closed subarcs of T, with $T_i \subseteq \text{domain}(f)$ and y_i an interior point of T_i. Let $T_i' = f(T_i)$. Since $y_i \in X^2$ there must be points $z_i' \in T_i' \cap X^1$

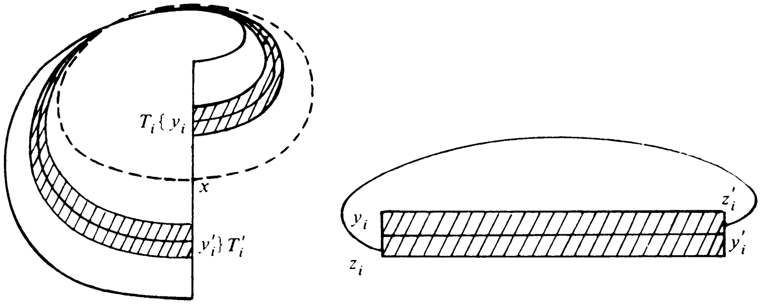

FIGURE 1

such that $z_i = f(z_i') \in T_i$. By Proposition 4 we may assume that all of the orbit segments $A_i = [y_i, y_i']$ are parallel, and that all the orbit segments $B_i = [z_i', z_i]$ are parallel. It follows that, for some n, the first return map on the subarc $[y_n', x]$ of T is monotone increasing. Note that, for i sufficiently large, z_i' is in the domain of this map (though y_i' need not be). As $z_i' \in X^1$, this contradicts Proposition 3.

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