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S-LIMITS AND $\mathcal{Q}$-SUMMABILITY

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ABSTRACT. Certain classes of subspaces of m are examined that are the domains of uniqueness of certain Banach type limits. These subspaces are identified as the bounded convergence domains of $\mathcal{Q}$-summability. As a corollary it is shown that the closure of bs in m is f_0. Also we characterize those matrices whose convergence domains include these spaces in a manner analogous to Lorentz' characterization of strongly regular matrices.

S-limits. Throughout this paper, S will denote a fixed regular matrix with $\|S\| = 1$. We shall limit our discussion to sequences and matrices of real numbers. A S-limit is a linear functional, L, on m, the space of bounded sequences, that satisfies: (i) $L(x) \geqslant 0$ if $x_n \geqslant 0, n = 1, 2, \ldots$; (ii) $L(\bar{1}) = \bar{1}$ ($\bar{1}$ is the constant sequence of 1's); (iii) $L(Sx) = L(x)$ for all $x \in m$. A Banach limit is a T-limit where T is the translation matrix, i.e. $(Tx)_n = x_{n+1}$. A matrix B is S-invariant if $B(S - I)$ maps every bounded sequence to a null sequence. (I is the identity matrix.)

S-limits need not exist for every regular matrix S. Let $S = (a_{nk})$ where $a_{nk} = 0$ if $k > 2n$ and $a_{nk} = 1$ if $k = 2n$. For $1 \leqslant k \leqslant 2n$, $a_{nk} = -1/n$ if k is odd, $a_{nk} = 1/n$ if k is even. S is regular, but for $x = \{1, 0, 1, 0, \ldots\}$, we have $Sx = \{-1, -1, -1, \ldots\}$. Now if L were an S-limit, we would have $0 \leqslant L(x) = L(Sx) = -1$. Thus S-limits need not exist.

THEOREM 1. *If there is a nonnegative regular S-invariant matrix, then S-limits exist.*

PROOF. Suppose B is such a matrix. Define a functional q on m by $q(x) = \operatorname{Lim\,sup} Bx$. Then q is sublinear and nonnegatively homogeneous. For $x \in c$ we have, by the regularity of B, that $\operatorname{Lim} x = q(x)$. By the Hahn-Banach Theorem, we can extend Lim to a linear functional, L, on m s.t. $L(x) \leqslant q(x)$ for all $x \in m$. Thus $-q(-x) \leqslant -L(-x) = L(x) \leqslant q(x)$, for all $x \in m$.

Since B is nonnegative, for $x_n \geqslant 0$, $n = 1, 2, \ldots$, we have $0 \leqslant \operatorname{Lim\,inf} Bx = -q(-x) \leqslant L(x)$. So L is nonnegative. Also for $x \in m$, $\operatorname{Lim} B(S - I)x = 0$; thus $-q(-(S - I)x) = 0 = q((S - I)x)$ which proves $L((S - I)x) = 0$. So L is an S-limit.

We note that the Cesàro matrix is a nonnegative regular T-invariant matrix;

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hence, as a corollary, we have that Banach limits exist.

For a particular S, the determination of the existence of an S-invariant matrix is facilitated by Schur's Theorem (see, for example, [8, p. 20]) which gives necessary and sufficient conditions for $B(S - I)$ to map m to n, the space of null sequences.

In the case where S-limits exist for a particular regular matrix S, we shall consider the following sets which are the domains of uniqueness of these limits:

$$\Lambda(S) = \{x \in m\colon L(x) = L'(x) \text{ for every pair of } S\text{-limits } L, L'\},$$

$$\Theta(S) = \{x \in m\colon L(x) = 0 \text{ for every } S\text{-limit } L\}.$$

Note that $\Lambda(S)$ and $\Theta(S)$ are closed subspaces of m since each S-limit L is nonnegative, hence continuous.

We shall identify $\Theta(S)$ and $\Lambda(S)$, in certain cases, as the bounded convergence domains of the method of summability introduced in [3].

$\mathfrak{A}$-summability. For $v = 1, 2, \ldots$, let $A^v = (a^v_{nk})$ be an infinite matrix of real numbers. Let $\mathfrak{A}$ denote the sequence of matrices $\{A^v\}$. A sequence $x = \{x_n\}$ is $\mathfrak{A}$ or $\{A^v\}$-summable to a number l if $\operatorname{Lim} A^v x = l$ uniformly in $v = 1, 2, \ldots$. We say $\mathfrak{A}x$ converges to l and write $\operatorname{Lim}_{\mathfrak{A}} x = l$. $\mathfrak{A}$ is said to be regular if $\mathfrak{A}x$ converges to l whenever x is convergent to l.

$$c_{\mathfrak{A}} = \{x\colon \mathfrak{A}x \text{ converges}\} \text{ and}$$

$$n_{\mathfrak{A}} = \{x\colon \mathfrak{A}x \text{ converges to } 0\}.$$

For a fixed regular matrix S, let Σ denote the sequence of matrices $\{S^0, S^1, S^2, \ldots\}$, where S^v denotes the product of S with itself v times. We shall use the notation $A \cdot \Sigma$ to denote the sequence of matrices $\{R^p\}$ where $(R^p x)_n = \Sigma a_{nk}(S^k x)_p$. (Throughout, all sums run from $k = 1$ to ∞ unless otherwise noted.) Note that for T, the translation matrix, $(T^k x)_p = (T^p x)_k$, so if $S = T$, $A \cdot \Sigma$ reduces to the sequence of ordinary matrix products of A and powers of T.

THEOREM 2. *Suppose A is regular. If A is T-invariant, then $A \cdot \Sigma$ is S-invariant, that is $A \cdot \Sigma(S - I)x$ converges to 0 for all $x \in m$.*

PROOF. Let $\mathfrak{R} = A \cdot \Sigma$. We show $\mathfrak{R}$ is S-invariant. For $x \in m$,

$$(R^p(S - I)x)_n = \Sigma a_{nk}(S^k(S - I)x)_p = \Sigma a_{nk}[(S^{k+1}x) - (S^k x)_p].$$

So

$$|(R^p(S - I)x_n| \leqslant |\Sigma a_{n,k-1}(S^k x)_p - \Sigma a_{nk}(S^k x)_p|$$

$$\leqslant \Sigma|a_{n,k-1} - a_{nk}|\,\|S^k\|\,\|x\|.$$

Since A is T-invariant, $\Sigma|a_{n,k-1} - a_{nk}| \to 0$ as $n \to \infty$ by Shur's Theorem.

Hence $(R^p(S - I)x)_n \to 0$ as $n \to \infty$ uniformly in p. This shows $A \cdot \Sigma$ is S-invariant.

Thus we have the following information. If A is a nonnegative regular matrix which is T-invariant and S-invariant, then Theorem 1 states S-limits exist and Theorem 2 states $A \cdot \Sigma$ is S-invariant.

THEOREM 3. *If A is regular, then $n_{A\cdot\Sigma} \cap m \subset \Theta(S)$.*

PROOF. Suppose $x \in n_{A\cdot\Sigma} \cap m$. Let $t_n = \{t_n^v\colon v = 1, 2, \ldots\}$ where $t_n^v = \Sigma a_{nk}(S^k x)_v$. Thus $t_n^v \to 0$ as $n \to \infty$ uniformly in $v = 1, 2, \ldots$, that is $t_n \to 0$ in $(m, \| \|_\infty)$. Let L be any S-limit. L is continuous on m so $L(t_n) \to 0$ as $n \to \infty$.

We shall now show $L(t_n)$ tends to $L(x)$. Now, for n fixed we have

$$\left| t_n^v - \sum_{k=1}^{N} a_{nk}(S^k x)_v \right| \leqslant \|x\| \sum_{k=N+1}^{\infty} |a_{nk}|\,\|S\|^k \leqslant \|x\| \sum_{k=N+1}^{\infty} |a_{nk}|$$

since $\|S\| = 1$. For each n, $\sum_{k=N+1}^{\infty} |a_{nk}| \to 0$ as $N \to \infty$ since A is regular. Thus $\sum_{k=1}^{N} a_{nk}(S^k x)_v \to t_n^v$ as $N \to \infty$ uniformly for $v = 1, 2, \ldots$ for each n. This implies, for an S-limit L, that

$$L(t_n) = \operatorname*{Lim}_{N\to\infty} \sum_{k=1}^{N} a_{nk} L(S^k x) = \operatorname*{Lim}_{N\to\infty} \sum_{k=1}^{N} a_{nk} L(x) = L(x) \sum_{k=1}^{\infty} a_{nk}$$

since L is continuous and S-invariant. Hence $\operatorname{Lim}_{n\to\infty} L(t_n) = L(x)$ since A is regular.

Therefore, since $L(t_n)$ tends to 0 and to $L(x)$, $x \in \Theta(S)$, which is the desired result.

THEOREM 4. *The closure of $(S - I)[m]$ in $(m, \| \|)$ is $\Theta(S)$.*

PROOF. Since an S-limit L is S-invariant, clearly $(S - I)[m] \subset \Theta(S)$. We shall show that every positive linear functional which is zero on $(S - I)[m]$ is also zero on $\Theta(S)$. (The positive linear functionals are total in the dual of $\Theta(S)$; see, e.g. [1, Theorem 4, p. 217].)

Suppose q is a positive linear functional on $\Theta(S)$ which vanishes on $(S - I)[m]$. We can extend q to a positive linear functional G on m with $G(\bar{1}) = 1$ (see, e.g., [5, p. 20]). Also $G((S - I)x) = 0$ for all $x \in m$, so G is S-invariant. Hence G is an S-limit, so G is zero on $\Theta(S)$. Therefore the closure of $(S - I)[m]$ is $\Theta(S)$.

COROLLARY 1. $\overline{bs} = f_0$.

PROOF. $bs = \{x\colon \sup_n |\sum_{k=1}^{n} x_k| < \infty\}$. It is easy to show $bs = (T - I)[m]$ where T is the translation matrix.

f_0 is the space of sequences that are almost convergent to zero which is precisely $\Theta(T)$.

The following result links S-limits to $A \cdot \Sigma$ summability.

THEOREM 5. *If A is regular and if $A \cdot \Sigma$ is S-invariant, then* $n_{A \cdot \Sigma} \cap m = \Theta(S)$.

PROOF. $A \cdot \Sigma$ is S-invariant, so $(S - I)[m] \subseteq n_{A \cdot \Sigma} \cap m$ and $n_{A \cdot \Sigma} \cap m \subseteq \Theta(S)$ from Theorem 3. Since $n_{A \cdot \Sigma} \cap m$ is closed in m and $\overline{(S - I)[m]} = \Theta(S)$, we have $n_{A \cdot \Sigma} \cap m = \Theta(S)$.

COROLLARY. *If, in addition,* $\bar{1} \in c_{A \cdot \Sigma}$, *then* $c_{A \cdot \Sigma} \cap m = \Lambda(S)$.

We note that $\bar{1} \in c_{A \cdot \Sigma}$ if $A \cdot \Sigma$ is regular or if A is regular and S is positive with all rows adding up to 1. A is said to be compatible to S if A is regular, $A \cdot \Sigma$ is a regular, S-invariant sequence of matrices. Thus if A is compatible to S, $c_{A \cdot \Sigma} \cap m = \Lambda(S)$. Necessary and sufficient conditions for compatibility can be formulated using Silverman-Toeplitz and Shur Theorems for $\mathfrak{A}$-summability found in [2].

Comparison theorems. We shall assume A is compatible to S. Let $\mathfrak{S} = A \cdot \Sigma$. Clearly if B were also compatible to S, then $c_{A \cdot \Sigma} \cap m = c_{B \cdot \Sigma} \cap m$. We show that among regular matrices the compatible ones are the "strongest".

THEOREM 6. *If B is regular, then* $n_{B \cdot \Sigma} \cap m \subset n_{A \cdot \Sigma} \cap m$ *if, in addition,* $1 \in c_{B \cdot \Sigma}$, $c_{B \cdot \Sigma} \cap m \subset c_{A \cdot \Sigma} \cap m$.

PROOF. Theorem 3 gives that $n_{B \cdot \Sigma} \cap m \subset \Theta(S)$ and compatibility gives that $n_{A \cdot \Sigma} \cap m = \Theta(S)$. This gives the desired conclusion.

If $1 \in c_{B \cdot \Sigma}$, since $A \cdot \Sigma$ is regular, we have $c_{B \cdot \Sigma} \cap m \subset c_{A \cdot \Sigma}$.

We say a matrix B is strongly $\mathfrak{A}$-regular if $Bx \to l$ whenever $\mathfrak{A}x \to l$. For $\mathfrak{T} = C \cdot \Gamma$ where C is a Cesàro matrix and $\Gamma = \{T^{\nu}\}$ the sequence of powers of T, strongly $\mathfrak{T}$-regular corresponds to strongly regular [6]. We have the following generalization of Lorentz' Theorem.

THEOREM 7. *B is strongly $\mathfrak{S}$-regular on m if and only if B is regular and S-invariant* (*or* $\operatorname{Lim}_{n \to \infty} \sum |(BS)_{nk} - B_{nk}| = 0$).

PROOF. Suppose B is $\mathfrak{S}$-regular on m. Then

$$n_B \cap m \supset n_{A \cdot \Sigma} \cap m \supset (S - I)[m].$$

So B is S-invariant. $A \cdot \Sigma$ is regular, so $x \to l$ implies $\mathfrak{S}x \to l$, thus $Bx \to l$. B is thus regular.

Conversely, B is S-invariant implies $n_B \cap m \supset (S - I)[m]$. $n_B \cap m$ is closed, so $n_B \cap m \supset \Theta(S)$. B is regular, thus $c_B \cap m \supset \Lambda(S)$; that B and $\mathfrak{S}$ are consistent follows from $n_B \cap B \supset \Theta(S)$.

As a consequence, we have a shortened proof of Lorentz' characterization of strongly regular matrices [6, Theorem 7].

COROLLARY. *B is strongly regular if and only if B is regular and* $\operatorname{Lim}_{n \to \infty} \sum |b_{nk} - b_{n,k+1}| = 0$.

PROOF. This is the above theorem applied to almost convergence, noting that $(BT)_{nk} = B_{n,k-1}$.

For $\mathcal{Q} = \{A^v\}$, consider the set $\mathfrak{U}$ of all matrices U where the nth row of U is the nth row of A^v for some v. Following along the idea of G. M. Petersen [7], it can be shown [2] that $\mathcal{Q}$ sums exactly those sequences that are summed by every member of $\mathfrak{U}$ to the same value.

The following generalizes a finding of R. J. Duran [4] for almost convergence.

THEOREM 8. $c_{\mathbb{S}} \cap m = \bigcap \{c_B \cap m$: *$B$ is regular and S-invariant*$\}$.

PROOF. If B is regular and S-invariant, it is strongly $\mathbb{S}$-regular on m, hence $c_{\mathbb{S}} \cap m \subset c_B \cap m$. Also, $c_{\mathbb{S}} = \bigcap \{c_U\colon U \in \mathfrak{U}\}$ where $\mathfrak{U}$ is described above. $c_{\mathbb{S}} \subset c_U$ are consistent implies that each U is strongly $\mathbb{S}$-regular and, hence, regular and S-invariant. Thus $\mathfrak{U}$ is a subcollection of all regular and S-invariant matrices, so $\bigcap \{c_U\colon U \in \mathfrak{U}\} \supset \bigcap \{c_B\colon B$ is regular and S-invariant$\}$, so $c_{\mathbb{S}} \cap m = \bigcap \{c_B \cap m\colon B$ is regular and S-invariant$\}$.

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