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SHORTER NOTES

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ON THE COMMUTATION THEOREM FOR TENSOR PRODUCTS OF VON NEUMANN ALGEBRAS

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ABSTRACT. It is shown that the commutation theorem for tensor products of general von Neumann algebras follows trivially from the case of von Neumann algebras with a separating and cyclic vector.

Let M_1 and M_2 be von Neumann algebras acting in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the commutation theorem for tensor products of von Neumann algebras states

$$(M_1 \otimes M_2)' = M'_1 \otimes M'_2.$$

The proof of this theorem in full generality was first obtained by Tomita in 1967 (see [3]). Later a number of simpler versions have been obtained (see e.g. [2]). Usually standard techniques are used to reduce the general case to the case of von Neumann algebras having a cyclic and separating vector. In this note we show that this reduction is almost a triviality.

Assume that the commutation theorem has been proved for von Neumann algebras with a separating and cyclic vector. To show that $(M_1 \otimes M_2)' = M'_1 \otimes M'_2$ it is sufficient to show that $(M_1 \otimes M_2)'$ and $(M'_1 \otimes M'_2)'$ commute since obviously $M'_1 \otimes M'_2 \subseteq (M_1 \otimes M_2)'$.

If $\omega_1 \in \mathcal{H}_1$ and $\omega_2 \in \mathcal{H}_2$ we define $e_1 = [M'_1\omega]$, $e'_1 = [M_1\omega]$ and similarly $e_2 = [M'_2\omega]$, $e'_2 = [M_2\omega]$ where $[M\omega]$ denotes the projection onto the closed subspace generated by $x\omega$ with $x \in M$. It is well known that $e_1 \in M_1$, $e'_1 \in M'_1$ and that the von Neumann algebras $e_1 M_1 e_1 e'_1$ and $e'_1 M'_1 e'_1 e_1$ as acting on $e_1 e'_1 \mathcal{H}$ are commutants of each other and have ω_1 as a separating and cyclic vector. A similar statement is true for M_2.

Denote

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad M = M_1 \otimes M_2, \quad N = M'_1 \otimes M'_2,$$
$$e = e_1 \otimes e_2 \quad \text{and} \quad e' = e'_1 \otimes e'_2.$$

From the commutation theorem for von Neumann algebras with a separating

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and cyclic vector applied to $e_1 M_1 e_1 e_1'$ and $e_2 M_2 e_2 e_2'$ we find that $eMee'$ and $e'Ne'e$ are commutants of each other on $ee'\mathcal{H}$. Now since $e \in M$ and $e' \in N \subseteq M'$ the commutant of $eMee'$ on $ee'\mathcal{H}$ is $e'M'e'e$. Similarly the commutant of $e'Ne'e$ on $ee'\mathcal{H}$ is $eN'ee'$. Hence $e'M'e'e$ and $eN'ee'$ commute.

Define $\omega = \omega_1 \otimes \omega_2$ and let $m' \in M'$ and $n' \in N'$. Using that $ee'\omega = \omega$ and that $e'm'e'e$ and $en'ee'$ commute, we obtain

$$\begin{aligned}\langle m'n'\omega, \omega\rangle &= \langle ee'm'n'ee'\omega, \omega\rangle = \langle (e'm'e)(e'n'e)\omega, \omega\rangle \\ &= \langle (e'm'e'e)(en'ee')\omega, \omega\rangle = \langle (en'ee')(e'm'e'e)\omega, \omega\rangle \\ &= \langle n'm'\omega, \omega\rangle.\end{aligned}$$

Using the polarization identity one can now obtain $m'n' = n'm'$. To see this denote $a = m'n' - n'm'$ so that $\langle a\omega_1 \otimes \omega_2, \omega_1 \otimes \omega_2\rangle = 0$ for all $\omega_1 \in \mathcal{H}_1$ and $\omega_2 \in \mathcal{H}_2$. For a fixed ω_2 consider the sesquilinear form $\phi(\xi_1, \eta_1) = \langle a\xi_1 \otimes \omega_2, \eta_1 \otimes \omega_2\rangle$ on $\mathcal{H}_1$. As $\phi(\omega_1, \omega_1) = 0$ for all $\omega_1 \in \mathcal{H}_1$ we get by polarization that $\phi = 0$, that means $\langle a\xi_1 \otimes \omega_2, \eta_1 \otimes \omega_2\rangle = 0$ for all $\xi_1, \eta_1 \in \mathcal{H}_1$ and $\omega_2 \in \mathcal{H}_2$. Similarly for fixed $\xi_1, \eta_1 \in \mathcal{H}_1$ using the sesquilinear form $\psi(\xi_2, \eta_2) = \langle a\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle$ on $\mathcal{H}_2$ we get that $\langle a\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle = 0$. But then $a = 0$. So $m'n' = n'm'$ for all $m' \in M'$ and $n' \in N'$. This completes the proof.

In [1] a simple proof of the commutation theorem is obtained already for the case of von Neumann algebras with a cyclic vector. Then of course the above simplification becomes even greater since it is sufficient to include only the projections e_1, e_2 and e and not e_1', e_2' and e'.

References

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