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ON THE STRUCTURE OF THE FIXED-POINT SET OF A NONEXPANSIVE MAPPING IN A BANACH SPACE

RONALD E. BRUCK, JR.

ABSTRACT. If C is a closed convex subset of a reflexive, strictly convex Banach space E, and $T: C \to E$ is a nonexpansive mapping which has a fixed-point in the interior of C, then there exists a nonexpansive mapping $T^*: E \to E$ whose fixed-point set in C is the fixed-point set of T.

In this note we investigate the structure of the fixed-point set $F(T)$ of a nonexpansive mapping $T: C \to E$, where C is a closed convex subset of a Banach space E and T does not necessarily map C into itself. It is known that T need not have an extension to a nonexpansive mapping $T^*: E \to \overline{\text{co}}\, T(C)$, (De Figueiredo and Karlovitz [3], Bruck [1]), so the following result is of interest:

THEOREM. *If C is a closed convex subset of a reflexive strictly convex Banach space E, $T: C \to E$ is nonexpansive, and T has a fixed point in the interior of C, then there exists a nonexpansive mapping $T^*: E \to E$ whose fixed points in C are exactly the fixed points of T.*

Before proving the theorem, we establish a variant of Lemma 5 of [2]:

LEMMA. *If y is a fixed point of T interior to C then there exists a nonexpansive retraction of E onto the cone*

$$K(y; F(T)) = \text{cl} \bigcup \{t \cdot F(T) + (1 - t)y: t > 0\}.$$

PROOF. Let $\delta > 0$ be so small that $B = \{x \in E: \|x - y\| \leqslant \delta\}$ is contained in C. Since $Ty = y$ and T is nonexpansive, $T(B) \subset B$. The restriction $T|_B$ is a nonexpansive mapping of B into itself, so by [2, Theorem 2] there exists a nonexpansive retraction r_1 of B onto $F(T|_B) = F(T) \cap B$. For $t > 0$ define

$$B_t = tB + (1 - t)y, \qquad F_t = t \cdot F(T) \cap B + (1 - t)y,$$

and

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$$r_t(x) = t \cdot r_1(t^{-1}x + (1 - t^{-1})y) + (1 - t)y$$

for $x \in B_t$, $r_t(x) = y$ for $x \in B_t$. $F(T)$ is convex because E is strictly convex, so for $0 < s < t$, $F_s \subset F_t$ and $B_s \subset B_t$. It is easy to verify that r_t is a retraction of E into F_t which is nonexpansive on B_t (but not on E). Evidently $\bigcup\{B_t: t > 0\} = E$ and cl $\bigcup\{F_t: t > 0\} = K(y; F(T))$. Denote $K(y; F(T))$ by K. We shall construct a nonexpansive retraction of E onto K as the limit–in an appropriate product space–of a subnet of $\{r_t: t > 0\}$ (limits taken as $t \to \infty$).

For x in E put $E_x = \{u \in E: \|u - y\| \leqslant \|x - y\|\}$, $P = \prod_{x \in E} E_x$, give E_x its weak topology, and P the corresponding product topology. By the reflexivity of E and Tychonoff's theorem, P is compact. Evidently $\{r_t: t > 0\}$ is a net in P, and therefore has a subnet which converges to some r in P.

Given any two x_1, x_2 in E, it follows from the weak lower semicontinuity of the norm, the nonexpansiveness of r_t on B_t, and the fact that $x_1, x_2 \in B_t$ for sufficiently large t, that $\|r(x_1) - r(x_2)\| \leqslant \|x_1 - x_2\|$. That is, r is nonexpansive. Since $F_s \subset F_t$ for $0 < s < t$, r_t leaves such point of F_s fixed; hence so does r. The denseness of $\bigcup F_s$ in K and the continuity of r imply that r fixes each point of K. Finally, each r_t maps E into $F_t \subset K$, and since K is weakly closed (being closed and convex) it follows that the range of r is contained in K. These three facts–r fixes each point of K, the range of r, is contained in K, and r is nonexpansive–mean r is a nonexpansive retraction of E onto K. Q.E.D.

Proof of Theorem. Put $R = \bigcap\{K(y; F(T)): y \in F(T) \cap \text{int } C\}$. By hypothesis, $F(T) \cap \text{int } C \neq \varnothing$, and by the Lemma, there exists a nonexpansive retraction of E onto each $K(y; F(T))$; by [2, Theorem 5], therefore, there exists a nonexpansive retraction of E onto R. Let T^* be such a retraction. We claim that T^* satisfies the conclusion of the Theorem. Obviously $F(T) \subset C \cap R = F(T^*|_C)$. If $F(T) \neq F(T^*|_C)$, let $x_0 \in C \cap R \backslash F(T)$. We reach a contradiction as follows: let $y_0 \in F(T) \cap \text{int } C$. Since $F(T)$ is closed, the intersection of the line segment $[x_0, y_0]$ with $F(T)$ contains a point z_0 closest to x_0. $z_0 \neq x_0$ (because $x_0 \notin F(T)$), $x_0 \in C$, and $y_0 \in \text{int } C$, therefore $z_0 \in \text{int } C$. Choose a point $x \neq z_0$ in $[x_0, z_0]$ which is closer to z_0 than to bdry C; thus $x \notin F(T)$. Now let y be the point of $F(T)$ which is closest to x (this exists because $F(T)$ is closed and convex and E is reflexive). Since x is closer to z_0 (which is in $F(T)$) than to bdry C, y must lie in int C. But R is convex, $x_0, z_0 \in R$, and $x \in [x_0, z_0]$, so $x \in R$. In particular,

$$x \in K(y; F(T)).$$

To summarize, there exists a point x such that $x \notin F(T)$, but for the point y of $F(T)$ closest to x, $x \in K(y; F(T))$. This is an obvious impossibility, and establishes $F(T^*|_C) = F(T)$. Q.E.D.

Corollary. *If, in addition to the hypotheses of the Theorem,* $F(T) \subset \text{int } C$, *then* $F(T^*) = F(T)$.

PROOF. $F(T^*)$ is convex and, by the Theorem, $F(T^*) \cap C = F(T) \subset \operatorname{int} C$; thus $F(T^*) \subset C$, hence $F(T^*) = F(T)$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007