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A NOTE ON KC WALLMAN COMPACTIFICATIONS

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ABSTRACT. In a previous paper, D. W. Hajek showed that if a space X is a T_3 space and A is a compact subset of WX, the Wallman compactification of X, then $X \cap A$ is a closed subset of X. This raises the question of whether this "closed intersection" property characterizes the T_3 spaces among the Hausdorff spaces. In the present paper, the authors show this conjecture is false by giving an example of a nonregular Hausdorff space whose Wallman compactification is a KC (compact closed)-space, and, hence, trivially satisfies this "closed intersection" property.

All spaces considered in this paper will be T_1 spaces. In [1] it was shown that if X is a T_3 space and if A is a compact subset of WX, the Wallman compactification of X, then $A \cap X$ is a closed subset of X. It is then natural to ask whether T_3 spaces are the only Hausdorff spaces which have this property. In this paper we give an example of a nonregular Hausdorff space whose Wallman compactification is a KC-space (i.e. is a space in which every compact subset is closed).

Recall that for any space X the Wallman compactification WX consists of the set $\{\mu: \mu$ is an ultrafilter in the lattice of all closed subsets of $X\}$ with topology generated by $\{C(A) = \{\mu \in : A \in \mu\} : A$ is a closed subset of $X\}$ as a base for the closed subsets of WX. With this topology WX is a compact T_1 space which is Hausdorff if and only if X is T_4, and the function $\varphi_X: X \to WX$ defined by $\varphi_X(x) = \{A : A$ is closed in X and $x \in A\}$ is a dense embedding of X in WX. When no ambiguity can result it is common practice to ignore the distinction between X and $\varphi_X[X]$ and to speak of X as a subset of WX. We note that it is easily shown that if A is a closed compact subset of X then $\mathrm{cl}_{WX}(A) = A$ and that for any closed subset A of X, $\mathrm{cl}_{WX}(A) = C(A)$.

We denote by T the space whose elements are the points in the unit interval $[0, 1]$ and whose topology is generated by the usual open subsets of $[0, 1]$ together with $[0, 1] \sim \{1/n: n$ a natural number$\}$. It is clear that T is a Hausdorff space and that any open set containing the closed set $\{1/n\}$ must intersect every neighborhood of 0; so T is not regular (see [3, p. 86]).

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THEOREM 1. *If A is a compact subset of WT then $A \cap T$ is a closed subset of T.*

PROOF. Suppose $A \cap T$ is not closed in T. Then $A \cap T$ has a limit point x in $T \sim A$. If $x \neq 0$ there is some number K such that $n > K$ implies that $(0, 1] \cap [x - 1/n, x + 1/n]$ is a closed compact subset of T, and, hence, of WT. This then implies that $\{A \cap [x - 1/n, x + 1/n]: n > K\}$ is a collection of closed subsets of A which has the finite intersection property and empty intersection. Since A is compact no such collection can exist; so A can have no limit points in T other than 0. We note that any element of $WT \sim T$ is clearly in $C(\{1/n\})$, and thus for any $\varepsilon > 0$ the set $[0, \varepsilon) \sim \{1/n\}$ is open both in T and in WT; so if 0 is a limit point of A, there must be a decreasing sequence $\{a_n\}$ of elements of A converging to 0, none of the elements of which are in $\{1/n\}$. It is clear, however, that if $0 \notin A$, then $\{a_n\}$ is a closed subset of A which is not compact.

With this theorem the following corollary is immediate from [2, Theorem 1].

COROLLARY. *If f: $X \to T$ is a Wallman extendible function, then the Wallman extension is unique.*

Theorem 1 tells us that the T_3 spaces are not the only spaces which have closed intersections with compact subsets of their Wallman compactifications. However we promised more.

THEOREM 2. *Every compact subset of WT is closed.*

PROOF. Suppose A is a compact subset of WT which is not closed. Then A must have a limit point μ which is not in A. From Theorem 1 we know that $\mu \notin T$, and again we note that the only elements of $WT \sim T$ are in $C(\{1/n\}) \sim \{1/n\}$. We also note that the relative topology on $\{1/n\}$ is the discrete topology. Since $\{C(E): E \in \mu, E \subseteq \{1/n\}\}$ is a collection of closed subsets of WT whose intersection is $\{\mu\}$, it is immediate that there is some subset B of $\{1/n\}$ such that $B \in \mu$ and $C(B)$ is disjoint from A. But since B is an infinite subset of a sequence, B itself must be a sequence. Then, as $A \cap T$ is closed in T, for each $x \in B$ there is an open subset $\mathfrak{U}_x$ of T such that $\mathfrak{U}_x$ is disjoint from A and $\mathfrak{U}_x \cap \{1/n\} = \{x\}$. It is then elementary to show that $C(B) \cup (\bigcup \mathfrak{U}_x)$ is an open subset of WT which contains μ and is disjoint from A.

We note that this process can be utilized in any compact Hausdorff space having a nonisolated first countable point to obtain the same results. It is not clear, however, whether a compact Hausdorff space with no first countable points must have a nonclosed subset which, when added to the closed sets yields a space with KC Wallman compactification or even the "closed intersection" property. The proof to Theorem 2 makes clear, however, that if a space is constructed by adding a set A to the closed subsets of a compact Hausdorff space and if the Wallman compactification of the subspace A is a KC-space, then the "closed intersection" property and having a KC Wallman compactification are equivalent.

We also note that [3, pp. 92–93] contains another example of a nonregular Hausdorff space which can be shown to have a KC Wallman compactification. Finally, the Euclidian plane with topology generated by the usual open sets together with all sets of the form $\{(0,0)\} \cup \mathcal{U}$ where $\mathcal{U}$ is obtained by deleting the positive y-axis from an open set of which it is a subset is an example of a Hausdorff space whose Wallman compactification does not have the "closed intersection" property.

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