PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

DEFORMATIONS OF GENERALIZED COMPLETE INTERSECTIONS

R. MANDELBAUM AND M. SCHAPS

ABSTRACT. The concept of a generalized complete intersection (GCI) of affine schemes is introduced. The proofs of the following theorems are then sketched.

THEOREM 1. *Suppose X is a GCI of affine Cohen-Macaulay schemes of codimension $\leqslant 2$. Then if the versal deformation space of X exists, it is smooth.*

THEOREM 2. *Suppose X is a GCI of determinantal schemes. Then if* $\dim X$ *is sufficiently low, X is smoothable. In particular, if X is as in Theorem* 1 *and* $\dim X < 4$ *then X is smoothable.*

Introduction. In this note we introduce the concept of a generalized complete intersection (GCI), that is, an intersection of schemes all of whose subintersections are of generic height. We then announce the result that (Theorem 1) the parameter space of the versal deformation space of a GCI of hypersurfaces and Cohen-Macaulay (C-M) schemes of codimension 2 is smooth. We also obtain (Theorem 2) that in sufficiently low dimension a GCI of determinantal schemes is smoothable, in particular if X is a GCI as in Theorem 1 and $\dim X < 4$ then X is smoothable. This generalizes results of ([Schaps], [Laksov]).

1. **Definitions.** Let $P = k[z_1, \ldots, z_q]$ be a polynomial ring in q variables over a fixed field k. We call an ideal J in P Cohen-Macaulay if P/J is a C-M ring of pure dimension.

DEFINITION 1. Let $J_1, \ldots, J_r$ be ideals in P and suppose for each subset $i_1, \ldots, i_s$ of $\{1, \ldots, r\}$ the ideal $J_{i_1} + \cdots + J_{i_s}$ is of height $\operatorname{ht} J_{i_1} + \cdots + \operatorname{ht} J_{i_s}$. Then we call $J = J_1 + \cdots + J_r$ a generalized complete intersection ideal (GCI).

We will see that (Lemma 1) if $J_1, \ldots, J_r$ are each complete intersections or C-M of codimension 2, and J is a GCI ideal, then $J_1, \ldots, J_r$ is a prime sequence in the following sense.

DEFINITION 2. A sequence of ideals $J_1, \ldots, J_r$ in P is a prime sequence iff $J_i \cap (J_1 + \cdots + J_{i-1}) = J_i(J_1 + \cdots + J_{i-1})$ for $i = 2, \ldots, r$.

We note that if the J_i are of ht 1 our definitions correspond to the standard definitions of a complete intersection ideal. We also note if the J_i are C-M then so is J.

Received by the editors July 14, 1975.
AMS (MOS) subject classifications (1970). Primary 14D15, 14B05.

2. **Preliminary lemmas.** Let J_1, J_2 be ideals in P such that J_1 and J_2 are Cohen-Macaulay and $J_1 + J_2$ is unmixed and $\operatorname{ht}(J_1 + J_2) = \operatorname{ht} J_1 + \operatorname{ht} J_2$. If J_1 is a hypersurface or $\operatorname{ht} J_1 = 2$ then

$$J_1 \cap J_2 = J_1 \cdot J_2.$$

PROOF. Clearly $J_1 \cap J_2 \supset J_1 \cdot J_2$ so it suffices to show that $J_1 \cap J_2 \subset J_1 \cdot J_2$.

Suppose J_1 is generated by $(f_1, \ldots, f_n)$. Let $d = \sum a_i f_i \in J_1 \cap J_2$ with $a_i \in P$. If $\operatorname{ht} J_1 = 1$ then f_1 is not a zero divisor in P/J_2, and thus $a_1 \in J_2$ and so therefore $d \in J_1 J_2$. So suppose $\operatorname{ht} J_1 = 2$ and thus by [Schaps] we may assume that $f_1, \ldots, f_n$ are the minors of an $n \times (n-1)$ relation matrix $R = (r_{ij})$ with columns $r_1, \ldots, r_{n-1}$. Then $d = \det [a, r_1, \ldots, r_{n-1}]$ where ${}^t a = (a_1, \ldots, a_n)$. Expanding d along its nth row we obtain $d = f_n a_n + b_1 r_{n1} + \cdots + b_{n-1} r_{n,n-1} \in J_2$ for appropriate b_i. Thus $f_n a + \sum b_i r_i \in J_2 P^n$. Again, multiplying by appropriate minors gives us $b_j J_1 \subset (f_n, J_2)$ for all j, and since J_1 is not contained in any of the associated primes of (f_n, J_2) we find $b_j \in (f_n, J_2)$. Thus $f_n(a + b'_j r_j) \in J_2 P^n$ and since f_n can be assumed to be prime to J_2 we obtain $a + \sum b'_j r_j \in J_2 P^n$, so $d \in J_1 J_2$.

LEMMA 2. *Let $J = J_1 + J_2$, $B = P/J$ and $B_i = P/J_i$. If $J_1 \cap J_2 = J_1 \cdot J_2$, then for any B-module,*

$$M \operatorname{Hom}_B (J/J^2, M) \simeq \operatorname{Hom}_{B_1} (J_1/J_1^2, M) \oplus \operatorname{Hom}_{B_2} (J_2/J_2^2, M).$$

PROOF. Use $J/J^2 \xrightarrow{\sim} J_1/J_1 J \oplus J_2/J_2 J \simeq (J_1/J_1^2 \otimes J_2/J_2^2) \otimes B$.

LEMMA 3. *Under the hypothesis of Lemma* 1

$$\operatorname{Hom}_{B_1} (J_1/J_1^2, B) \xrightarrow{\sim} \operatorname{Hom}_{B_1} (J_1/J_1^2, B_1) \otimes_{B_1} B.$$

PROOF. We use techniques similar to those used in the proof of Lemma 1.

LEMMA 4. *Suppose J_1, J_2 are ideals such that $J_1 \cap J_2 = J_1 J_2$. Then any flat deformation of J_1 and J_2 over an Artin or complete local ring R lifts to a flat deformation of $J = J_1 + J_2$.*

PROOF. A deformation over R is flat iff all the relations can be lifted. The relations on a generating set of $J_1 + J_2$ are generated by relations on J_1, relations on J_2 and trivial relations, all of which are liftable.

3. **Main theorems.**

THEOREM 1. *Suppose $J_1, \ldots, J_r$ are either hypersurfaces or C-M of codimension 2 and $J = J_1 + \cdots + J_r$ is a generalized complete intersection ideal. Then if the versal deformation space of $X = \operatorname{Spec}(P/J)$ exists, it has a smooth parameter space.*

PROOF. Let $I_t = J_1 + \cdots + J_t$, $B_t = P/J_t$, $C_t = P/I_t$ and $B = P/J$. Then by Lemma 1, $I_t \cap I_{t+1} = I_t \cdot I_{t+1}$ and by Lemma 2,

$$\operatorname{Hom}_B(J/J^2, B) \simeq \bigoplus_{t=1}^{r} \operatorname{Hom}_{B_t}(J_t/J_t^2, B).$$

By Lemmas 1 and 3 we get

$$\operatorname{Hom}_{B_t}(J_t/J_t^2, B) \simeq \operatorname{Hom}_{B_t}(J_t/J_t^2, B_t) \oplus {}_{B_t}B.$$

Thus $\operatorname{Hom}_B(J/J^2, B)$ is generated by the generators of the various $\operatorname{Hom}_{B_t}(J_t/J_t^2, B_t)$. Now (by [Schlessinger]), X has a versal deformation space iff

$$t_B = \operatorname{Hom}_B(J/J^2, B)/\operatorname{Im}(\operatorname{Hom}_B(\Omega_{P/k} \otimes B, B))$$

is finite dimensional as a k-vector space (where $\Omega_{P/k}$ and the map

$$\operatorname{Hom}_B(\Omega_{P/k} \otimes B, B) \to \operatorname{Hom}(J/J^2, B)$$

are as in [Schlessinger]).

We can take a k-basis $x_1, \ldots, x_N$ for t_B consisting of multiples of the generators over B of the various $\operatorname{Hom}_{B_t}(J_t/J_t^2, B)$. Letting $T_1, \ldots, T_N$ be indeterminates, the generic first order deformation over $k[T]/(T)^2$ induces deformations of each of the J_t, each of which has a flat lifting over $k[[T]]$. Therefore, by Lemma 4, the deformation of J can be lifted over $k[[T]]$, giving the versal deformation space, with smooth parameter space Spec$[[T]]$.

Using techniques of [Laksov] we obtain

THEOREM 2. *Suppose $P = k[z_1, \ldots, z_q]$ with k a fixed field.*

Let $X_1, \ldots, X_m$ be determinantal schemes with $X_t = \operatorname{Spec}(P/J_t)$ and J_t generated by the $l_t \times l_t$ minors of an $m_t \times n_t$ matrix. Put

$$X = \operatorname{Spec}(P/J_1 + \cdots + J_m)$$

and suppose that X has pure codimension equal to $\sum \operatorname{codim}(X_t)$.

Then if X is a generalized complete intersection and if

$$\dim X < \min_{t \in N}(m_t + n_t - 2l_t + 3)$$

where $N = \{t \mid X_t$ is not a complete intersection$\}$ then X is smoothable.

COROLLARY. *Suppose each X_i above is either a complete intersection or C-M of codimension 2. Then $\dim X < 4$ implies X is smoothable.*

REFERENCES

D. Laksov, *Deformation of determinantal varieties*, Univ. of Oslo (to appear).

M. Schaps, *Deformations of Cohen-Macaulay schemes of codimension* 2 *and non-singular deformations of space curves* (to appear).

M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222. MR **36** #184.

WEIZMANN INSTITUTE OF SCIENCE, P.O.B. 26, REHOVOT, ISRAEL

MATHEMATICAL SCIENCES DEPARTMENT, TEL-AVIV UNIVERSITY, TEL-AVIV, ISRAEL (Current address of M. Schaps)

Current address (R. Mandelbaum): School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540