

REPRESENTATIONS OF COMPACT GROUPS ON TOPOLOGICAL VECTOR SPACES: SOME REMARKS

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ABSTRACT. A standard theorem concerning the decomposition of a representation of a compact group on a Hilbert space E is generalized to the case when E is locally convex and quasi-complete. As a corollary, it is shown that if E is topologically irreducible, then it is finite dimensional.

1. **Introduction.** Let a compact group G be represented unitarily on a Hilbert space H . Then there is a "canonical" decomposition $H = \bigoplus_{\sigma \in \hat{G}} H_{\sigma}$, where the invariant subspaces H_{σ} are indexed by the elements σ of \hat{G} , the set of equivalence classes of irreducible unitary representations of G . Each subspace H_{σ} is characterized as the smallest subspace of H containing all invariant subspaces V of type σ (V is of type σ if the representation of G on V is in σ). Each H_{σ} may be expressed (nonuniquely) as a direct sum of invariant subspaces $H_{\sigma,i}$ of type σ .

We here replace H by a quasi-complete topological vector space E . Analogues of the above statements are proven. It is also shown that if E is topologically irreducible, then E has finite dimension.

2. Preliminaries.

2.1. *Notation.* We let E denote a Hausdorff, locally convex, quasi-complete (closed bounded sets are complete) topological vector space, G a compact topological group which is represented continuously on E . Thus there is a continuous map $(g, e) \rightarrow g \cdot e: G \times E \rightarrow E$ such that each map $g: e \rightarrow g \cdot e$ is linear. The notation (G, E) indicates that G is represented continuously on E . We use dg, dg' , etc. to refer to normalized Haar measure on G .

2.2. **DEFINITIONS.** Let \hat{G} be the set of equivalence classes of irreducible unitary representations of G [2, 27.3]. If $\sigma \in \hat{G}$, let d_{σ} be the dimension of V for some (hence any) $(G, V) \in \sigma$.

2.3. **DEFINITION.** A representation (G, E) is *topologically irreducible* if E contains no proper closed G -invariant subspaces except $\{0\}$.

We review the definition and certain basic properties of the weak integral; for more details see [1, Chapter III, §§3–4].

2.4. Let X be a locally compact topological space. Let $K(X) = \{f: X \rightarrow \mathbb{C} \mid f \text{ is continuous with compact support}\}$, $K(X, E) = \{f: X \rightarrow E \mid f \text{ is continuous with compact support}\}$. Let E' be the *topological dual* of E (*continu-*

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ous linear functionals), E^* the algebraic dual of E' (all linear functionals). Let μ be a measure on X (a continuous linear functional on $(K(X), \mathfrak{T})$, \mathfrak{T} = topology of uniform convergence on compact sets). If $f \in K(X, E)$, define the weak integral of f with respect to μ as follows:

$$\left\langle e', \int_X f(x) d\mu(x) \right\rangle = \int_X \langle e', f(x) \rangle d\mu(x) \quad (e' \in E').$$

2.5. THEOREM. (a) If $f \in K(X, E)$, then $\int_X f(x) d\mu(x) \in E$.

(b) If q is a continuous seminorm on E , then $q(\int_X f(x) d\mu(x)) \leq \int_X q(f(x)) d\mu(x)$.

(c) If Y is another locally compact space, ν a measure on Y , and

$$f \in K(X \times Y, E),$$

then

$$\begin{aligned} \int_{X \times Y} f(x, y) d\mu(x) d\nu(y) &= \int_X d\mu(x) \left[\int_Y f(x, y) d\nu(y) \right] \\ &= \int_Y d\nu(y) \left[\int_X f(x, y) d\mu(x) \right]. \end{aligned}$$

For the proofs see [1, Chapter III: §3, No. 3, Corollary 2; §3, No. 2, Proposition 6; §3, No. 2, Proposition 2; §4, No. 1, Remark following Theorem 2].

2.6. Let $M(G)$ be the convolution algebra [2, 19.12] of G ; denote convolution by $*$. The action of G on E extends to an algebra homomorphism U from $M(G)$ to $L(E)$, the continuous linear operators on E ; the extension is given by $U(v) \cdot e = \int_G (g \cdot e) dv(g)$. For a proof of this, see the two paragraphs preceding 4.1.1.2 in [3]; note that (i) G is compact, so all measures on G have compact support; (ii) the proof uses only quasi-completeness, not completeness, of E .

2.7. DEFINITIONS. Let $\sigma \in \hat{G}$, $(G, V) \in \sigma$. Let v_1, \dots, v_n ($n = d_\sigma$) be an orthonormal basis in V , and let $u_{ij}^\sigma(g) = \langle gv_j, v_i \rangle$ ($g \in G, 1 \leq i, j \leq n$), where \langle, \rangle is the inner product in V . Let $\mathfrak{T}_\sigma(G)$ be the linear span in $C(G)$ of the functions u_{ij}^σ , and let $\mathfrak{T}(G)$ be the span in $C(G)$ of $\cup_{\sigma \in \hat{G}} \mathfrak{T}_\sigma(G)$. Let $\lambda_\sigma(g) = \sum_{i=1}^n u_{ii}^\sigma(g)$ ($g \in G$); λ_σ is the character of σ . See [2, 27.5–27.8].

2.8. THEOREM. (a) (Orthogonality) $\int_G u_{ij}^\sigma(g) u_{kl}^\tau(g) dg = \delta_{ik} \delta_{jl} \delta_{\sigma\tau}$ ($\sigma, \tau \in \hat{G}$).

(b) (Peter-Weyl theorem) $\mathfrak{T}(G)$ is uniformly dense in $C(G)$.

(c) If $\sigma, \tau \in \hat{G}$ and $\phi \in \mathfrak{T}_\sigma(G)$, then $\phi * d_\sigma \lambda_\sigma = d_\sigma \lambda_\sigma * \phi = \phi$.

(d) If $\sigma, \tau \in \hat{G}$, then $\lambda_\sigma * \lambda_\tau = \delta_{\sigma\tau} \lambda_\sigma / d_\sigma$.

For the proofs, see [2, 27.19; 27.40; 27.24(ii); 27.24(iii)].

2.9. DEFINITION. Let (G, E) be a representation, E quasi-complete. Say that (G, E) is of type σ if for some (hence any) $(G, V) \in \sigma$, there is a linear isomorphism $W: V \rightarrow E$ such that $W(g \cdot v) = g \cdot W(v)$. (Thus W intertwines (G, E) and (G, V) .) Abusing notation, we write $(G, E) \in \sigma$.

3. Results. Recall that a locally convex topology is determined by its set of continuous seminorms.

3.1. DEFINITION. A seminorm p on E is G -invariant if $p(g \cdot e) = p(e)$ ($g \in G, e \in E$).

3.2. LEMMA. *The topology on E is determined by the set of continuous G -invariant seminorms.*

PROOF. Let q be a continuous seminorm on E . It suffices to show that there is a continuous G -invariant seminorm p such that $q(e) \leq p(e)$ ($e \in E$). To prove this, let $p(e) = \sup_{g \in G} q(ge)$; p is an invariant seminorm. To demonstrate continuity of p , let $e_n \rightarrow 0$ (a seminorm on E is continuous iff it is continuous at 0). Let $\epsilon > 0$ be given, and let $U_\epsilon = \{e \in E | q(e) < \epsilon\}$. Since G is compact, it acts equicontinuously on E ; let V be a neighborhood of 0 such that $G \cdot V \subset U_\epsilon$. If $n > n_0 \Rightarrow e_n \in V$, then $n \geq n_0 \Rightarrow p(e_n) = \sup_{g \in G} q(ge_n) < \epsilon$. So p is continuous.

Let $\sigma \in \hat{G}$; we define an operator $P_\sigma: E \rightarrow E$ as follows:

$$P_\sigma(e) = d_\sigma \int_G \overline{\lambda_\sigma(g)}(ge) dg,$$

where λ_σ is the appropriate character (2.7). By 2.5(a), P_σ is well defined; it is easily seen to be linear.

3.3. PROPOSITION. (a) P_σ is continuous ($\sigma \in \hat{G}$).

(b) $P_\sigma P_\tau = P_\tau P_\sigma = 0$ ($\sigma \neq \tau$); $P_\sigma^2 = P_\sigma$.

(c) $gP_\sigma = P_\sigma g$ ($g \in G$).

PROOF. In the proof, we make free use of 2.5(a)–(d).

(a) Let $e_n \rightarrow 0$ and suppose p is a continuous G -invariant seminorm. Then

$$p(P_\sigma(e_n)) = d_\sigma p\left(\int_G \overline{\lambda_\sigma(g)}(ge_n) dg\right) \leq d_\sigma \int_G |\overline{\lambda_\sigma(g)}| p(ge_n) dg \leq d_\sigma^2 p(e_n) \rightarrow 0.$$

By 3.1, P_σ is continuous.

(b) Note that

$$\begin{aligned} P_\sigma(P_\tau(e)) &= d_\sigma^2 \int_G \overline{\lambda_\sigma(g)} g \cdot \left[\int_G \overline{\lambda_\tau(g')}(g'e) dg' \right] dg \\ &= d_\sigma^2 \int_G \int_G \overline{\lambda_\tau(g)} \overline{\lambda_\sigma(g')}(gg'e) dg' dg \\ &= d_\sigma^2 \int_G \tilde{g}e \left[\int_G \overline{\lambda_\sigma(g)} \overline{\lambda_\tau(g^{-1}\tilde{g})} dg \right] \\ &= d_\sigma^2 \int_G (\tilde{g}e) \overline{\lambda_\sigma * \lambda_\tau(\tilde{g})} d\tilde{g}. \end{aligned}$$

Since $\lambda_\sigma * \lambda_\tau = \delta_{\sigma\tau} \lambda_\sigma / d_\sigma$, the last integral is zero if $\sigma \neq \tau$, and is $d_\sigma \int_G (\tilde{g}e) \overline{\lambda_\sigma(\tilde{g})} d\tilde{g} = P_\sigma(e)$ if $\sigma = \tau$.

(c) We compute:

$$\begin{aligned}
 g \cdot P_\sigma(e) &= d_\sigma g \cdot \int_G \overline{\lambda_\sigma(g')}(g'e) dg' = d_\sigma \int_G \overline{\lambda_\sigma(g')}(gg'e) dg' \\
 &= d_\sigma \int_G \overline{\lambda_\sigma(g^{-1}\tilde{g})}(\tilde{g}e) d\tilde{g} = d_\sigma \int_G \overline{\lambda_\sigma(\tilde{g}g^{-1})}(\tilde{g}e) d\tilde{g} \\
 &= d_\sigma \int_G \overline{\lambda_\sigma(g')}(g'ge) dg' = P_\sigma(ge).
 \end{aligned}$$

In 2.6, we defined the algebra homomorphism $U: M(G) \rightarrow L(E)$.

Let $E_\sigma = P_\sigma(E)$ ($\sigma \in \hat{G}$). From 3.3(b) and (c), we see that $E_\sigma \cap E_\tau = \{0\}$ ($\sigma \neq \tau$), and $G \cdot E_\sigma = E_\sigma$.

3.4. PROPOSITION. (a) E_σ is closed.

(b) The weak direct sum $\bigoplus_{\sigma \in \hat{G}} E_\sigma$ is dense in E .

(c) $P_\sigma(e) = 0$ for all $\sigma \in \hat{G}$ iff $e = 0$.

PROOF. (a) If (e_n) is a net in E_σ with $e_n \rightarrow e \in E$, then $P_\sigma(e) = \lim_n P_\sigma(e_n) = \lim_n e_n = e$.

(b) Write $\overline{\mathfrak{F}_\sigma(G)}$ for $\{\bar{\phi} | \phi \in \mathfrak{F}_\sigma(G)\}$. It follows from 2.8(c) that, if $\phi \in \overline{\mathfrak{F}_\sigma(G)}$, then $\phi * d_\sigma \bar{\lambda}_\sigma = d_\sigma \bar{\lambda}_\sigma * \phi = \phi$. Since U is a $*$ -homomorphism, one has $U(\phi)P_\sigma = P_\sigma U(\phi) = U(\phi)$ (we used the fact that $U(d_\sigma \bar{\lambda}_\sigma) = P_\sigma$, which is immediate from the definitions). We now reason as follows. Let $S = \text{cls} \bigoplus_{\sigma \in \hat{G}} E_\sigma$, and let $\eta \in E'$ be such that $\eta|_S = 0$. Fix $\sigma \in \hat{G}$ and $e \in E$; then for all $\phi \in \overline{\mathfrak{F}_\sigma(G)}$, one has $P_\sigma U(\phi)e \in S$, so

$$0 = \langle P_\sigma U(\phi) \cdot e, \eta \rangle = \langle U(\phi)e, \eta \rangle = \int_G \langle ge, \eta \rangle \phi(g) dg.$$

Let σ vary over \hat{G} ; then ϕ may be any element of $\bigcup_{\sigma \in \hat{G}} \overline{\mathfrak{F}_\sigma(G)}$, which has dense linear span in $C(G)$ (2.8(b)). It follows that $\langle ge, \eta \rangle = 0$ for all $g \in G$. Let $g = \text{id}$ to see that $\eta = 0$. Since E is locally convex, $S = E$.

(c) Suppose $P_\sigma(e) = 0$ ($\sigma \in \hat{G}$). Then $0 = U(\phi)P_\sigma e = U(\phi)e$ ($\phi \in \overline{\mathfrak{F}_\sigma(G)}$). Repeating part of the proof of (b) shows that $\langle e, \eta \rangle = 0$ for each $\eta \in E^*$, so $e = 0$.

Recalling 2.7, pick $(G, V) \in \sigma$. Let v_1, \dots, v_n ($n = d_\sigma$) be an orthonormal basis for V , and let $\{u_{ij} | 1 \leq i, j \leq n\}$ be the corresponding coordinate functions.

3.5. LEMMA. Let $0 \neq e \in E_\sigma$. Then there exist finitely many invariant subspaces $(B_j)_{j=1}^l$, $1 \leq l \leq d_\sigma$, $(G, B_j) \in \sigma$, such that $e \in \sum_{j=1}^l B_j$ (the sum, not the direct sum).

PROOF. We have $\lambda_\sigma(g) = \sum_{j=1}^n u_{jj}(g)$. Hence one has $e = P_\sigma(e) = d_\sigma \sum_{j=1}^n U(\bar{u}_{jj}) \cdot e$. Let $J \subset \{1, 2, \dots, n\}$ be those integers such that $U(\bar{u}_{jj})e \neq 0$. Fixing $j \in J$, we will show that $\{U(\bar{u}_{ij})e | 1 \leq i \leq n\}$ spans a subspace B_j of the desired type.

We first show that if $g \in G$, then

$$(*) \quad g \cdot U(\bar{u}_{rs}) = \sum_{p=1}^j u_{pr}(g) U(\bar{u}_{ps}).$$

To see this, let $a \in E$, $\eta \in E^*$, and let $g': E' \rightarrow E''$ be the adjoint of g . Then

$$\begin{aligned} \langle g \cdot U(\bar{u}_{rs})a, \eta \rangle &= \langle U(\bar{u}_{rs})a, g'\eta \rangle = \int_G \langle g'a, g'\eta \rangle \bar{u}_{rs}(g') dg' \\ &= \int_G \langle gg'a, \eta \rangle \bar{u}_{rs}(g') dg' = \int_G \langle g'a, \eta \rangle \bar{u}_{rs}(g^{-1}g') dg'. \end{aligned}$$

Now

$$\bar{u}_{rs}(g^{-1}g') = \sum_{p=1}^n \bar{u}_{rp}(g^{-1})\bar{u}_{ps}(g'),$$

so the equality continues:

$$\sum_{p=1}^n u_{pr}(g) \int_G \langle g'a, \eta \rangle \bar{u}_{ps}(g') dg' = \sum_{p=1}^n u_{pr}(g) \langle U(\bar{u}_{ps}) \cdot a, \eta \rangle.$$

Comparing first and last terms show that (*) holds.

Recall $j \in J$ is fixed. Let $e_i = U(\bar{u}_{ij}) \cdot e$ ($1 \leq i \leq n$). Since $U(\bar{u}_{ij})P = P_\sigma U(\bar{u}_{ij})$, $e_i \in E$. Define $W: V \rightarrow B_j = \text{span}\{e_1, \dots, e_n\}$ by $W(v_i) = e_i$ and by linearity. We compute:

$$\begin{aligned} W(gv_i) &= W\left(\sum_{p=1}^n u_{pi}(g)v_p\right) = \sum_{p=1}^n u_{pi}(g)e_p \\ &= \sum_{p=1}^n u_{pi}(g)U(\bar{u}_{pj}) \cdot e = g \cdot U(\bar{u}_{ij})(e) \end{aligned}$$

by (*). The last term is $g \cdot e_i = gW(v_i)$; we conclude that B_j is G -invariant and that W intertwines (G, V) and (G, B_j) . Since $e_j \neq 0$, $(G, B_j) \in \sigma$.

Apply the above analysis to each $j \in J$. We obtain subspaces $B_j \subset E_\sigma$ such that $(G, B_j) \in \sigma$. Any two of these subspaces either coincide or intersect in $\{0\}$ by irreducibility. Form the sum $B = \sum_{j \in J} B_j$; then $e = \sum_{j \in J} U(\bar{u}_{jj}) \cdot e \in B$.

Lemma 3.5 shows that if $E_\sigma \neq 0$, then E contains "many" irreducible subspaces. We can restate this as follows.

3.6. PROPOSITION. *If $E_\sigma \neq 0$, there is a collection $(B_i)_{i \in I}$ of G -invariant subspaces of E_σ , with $(G, B_i) \in \sigma$, such that $\sum_{i \in I} B_i = E_\sigma$ (the weak sum, not its closure!).*

PROOF. Let M be the set of all collections $(B'_i)_{i \in I'}$ of distinct subspaces of E_σ satisfying $(G, B'_i) \in \sigma$; partially order M by inclusion. By Lemma 3.5, $M \neq \Phi$. Clearly we can zornify; let $(B_i)_{i \in I}$ be maximal in M . Since the B_i are distinct and irreducible, $B_i \cap B_j = \{0\}$ ($i \neq j$). We now define $B = \sum_{i \in I} B_i$. If $e \in E_\sigma$, $e \notin B$, perform the construction of 3.5 to obtain subspaces C_j , $1 \leq j \leq l$, such that $(G, C_j) \in \sigma$. At least one C_{j_0} is not in $(B_i)_{i \in I}$, for otherwise $e \in \sum_{j=1}^l C_j \subset \sum_{i \in I} B_i$. Add C_{j_0} to $(B_i)_{i \in I}$ to obtain a contradiction.

3.7. PROPOSITION. E_σ is the smallest closed subspace of E containing all G -invariant subspaces B such that $(G, B) \in \sigma$.

PROOF. Let $B \subset E$, $(G, B) \in \sigma$, and let $V, v_1, \dots, v_n, u_{ij}$ be as in 3.5. Let $W: V \rightarrow B$ be intertwining, $e_i = W(v_i)$. Then

$$\begin{aligned} P_\sigma(E_i) &= d_\sigma \int_G (ge_i) \overline{(\lambda_\sigma(g))} dg = d_\sigma \int_G g \cdot W(v_i) \overline{\lambda_\sigma(g)} dg \\ &= d_\sigma \int_G W(gv_i) \overline{\lambda_\sigma(g)} dg = W \left(\int_G (gv_i) \overline{\lambda_\sigma(g)} dg \right). \end{aligned}$$

The last step uses 2.5(c) and the fact that W is continuous as a map from V to E . The equality continues:

$$\begin{aligned} &= W \left[d_\sigma \int_G \left(\sum_{p=1}^n u_{pi}(g) v_p \right) \cdot \left(\sum_{r=1}^n \overline{u_{rr}(g)} \right) dg \right] \\ &= W \left[d_\sigma \sum_{p=1}^n \left(\int_G \sum_{r=1}^n u_{pi}(g) \overline{u_{rr}(g)} dg \right) v_p \right] \\ &= W \left[d_\sigma \sum_{p=1}^n \frac{1}{d_\sigma} \delta_{pi} v_p \right] = W(v_i) = e_i. \end{aligned}$$

We have used the orthogonality results for the u_{ij} 's (2.8(a)). This shows that $P_\sigma e_i = e_i$; i.e., $e_i \in E$. Thus $B \subset E_\sigma$, so E_σ contains all invariant subspaces such that $(G, B) \in \sigma$. Lemma 3.5 shows that E_σ is the smallest such.

3.8. REMARK. The proofs of 3.4, 3.5, and 3.7 are straightforward generalizations of portions of the proof of 27.44 in [2].

3.9. THEOREM. Let $E \neq \{0\}$ be topologically irreducible. Then $(G, E) \in \sigma$ for some $\sigma \in \hat{G}$. In particular, E has finite dimension.

PROOF. Since $E \neq \{0\}$, there exists $\sigma \in \hat{G}$ such that $P_\sigma \neq 0$ (3.4(c)). Consider one such σ ; the space E_σ is invariant and closed, hence $E_\sigma = E$. Let $e \in E_\sigma$; by 3.5, $e \in \bigoplus_{j=1}^l B_j$, where each $(G, B_j) \in \sigma$. By finite dimensionality $\bigoplus_{j=1}^l B_j$ is closed; it follows easily that $l = 1$ and that $(G, E) \in \sigma$.

3.10. COROLLARY. Let $E' \subset E$ be topologically irreducible. Then $E' \subset E_\sigma$ for some $\sigma \in \hat{G}$.

PROOF. Since $(G, E') \in \sigma$ for some σ (3.9), $E' \subset E_\sigma$ (3.7).

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