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A SUFFICIENT CONDITION FOR HYPERINVARIANCE

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ABSTRACT. A linear transformation on a finite-dimensional complex linear space has the property that all of its invariant subspaces are hyperinvariant if and only if its lattice of invariant subspaces is distributive [1]. It is shown that an operator on a complex Hilbert space has this property if its lattice of invariant subspaces satisfies a certain distributivity condition.

1. **Preliminaries**. Throughout this paper H will denote an arbitrary complex Hilbert space. All operators are bounded and all subspaces are closed. By a *subspace lattice* on H is meant a family of subspaces of H which is closed under the formation of arbitrary intersections and arbitrary closed linear spans and which contains the zero subspace (0) and H. The family of subspaces invariant under an operator T is denoted by Lat T. This is a subspace lattice as is the family of subspaces invariant under every operator commuting with T which we denote by Hyperlat T. The elements of Hyperlat T are called the *hyperinvariant* subspaces of T. Clearly Hyperlat $T \subseteq$ Lat T. A subspace lattice $\mathcal{F}$ is called *commutative* if for every pair of subspaces M, $N \in \mathcal{F}$ the corresponding projections P_M and P_N commute. Let L be an abstract lattice. We say that L is

(i) *distributive* if

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad (a, b, c \in L)$$

and its dual statement holds identically;

(ii) *σ-infinitely meet distributive* if L is σ-complete and

$$a \vee \{\textstyle\bigwedge b_n : n \geqslant 1\} = \textstyle\bigwedge \{a \vee b_n : n \geqslant 1\} \qquad (a, b_n \in L)$$

holds identically in L.

That the dual equation defining distributivity are equivalent to each other is an elementary result of lattice theory.

2. **A sufficient condition for hyperinvariance**. The key to the sufficient condition is a result of Sarason and the following lattice-theoretic result.

PROPOSITION 2.1. *If L is an abstract σ-infinitely meet distributive lattice and $\theta: L \to L$ is a lattice automorphism with the properties*

(I) $a \leqslant \theta(a) \vee \theta^{-1}(a)$ $(a \in L)$;

(II) a, $\theta(a)$ *comparable implies* $a = \theta(a)$,

then θ is the identity automorphism.

PROOF. For every $n \geqslant 1$ let $\theta^n: L \to L$ be defined in the obvious way. Let a

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be an arbitrary fixed element of L. For $n \geqslant 1$ put $a_n = a \wedge \theta(a) \wedge \theta^2(a) \wedge \cdots \wedge \theta^n(a)$. Then $a_n \wedge \theta(a_n) = a_{n+1}$ $(n \geqslant 1)$. Using (I) and the fact that L is distributive the statement

$$\theta(x) = (x \wedge \theta(x)) \vee \left[\theta(x \wedge \theta(x))\right]$$

holds identically in L. Using this identity it is easily shown that $\theta(a) = a_1 \vee \theta(a_n)$ $(n \geqslant 1)$. Thus

$$\theta(a) = \bigwedge\{a_1 \vee \theta(a_n) : n \geqslant 1\} = a_1 \vee \left(\bigwedge\{\theta(a_n) : n \geqslant 1\}\right).$$

If $c = \bigwedge\{a_n : n \geqslant 1\}$ then

$$\theta(c) = \bigwedge\{\theta(a_n) : n \geqslant 1\} \geqslant \bigwedge\{a_{n+1} : n \geqslant 1\} \geqslant c$$

and by (II), $\theta(c) = c$. Hence $\theta(a) = a_1 \vee \theta(c) = a_1 \vee c = a_1 \leqslant a$ and again by (II), $\theta(a) = a$. This completes the proof.

Let T be an operator on H. Notice that if S is an invertible operator commuting with T then $SM \in \operatorname{Lat} T$ $(M \in \operatorname{Lat} T)$ and the mapping $M \to SM$ is a lattice automorphism with the mapping $M \to S^{-1}M$ as its inverse. If the operator A commutes with T and μ is a scalar with $|\mu| > \|A\|$, the operator $S = A - \mu$ is invertible, commutes with T and $\operatorname{Lat} A = \operatorname{Lat} S$. By a result of Sarason [4], $\operatorname{Lat} S = \operatorname{Lat} S^{-1}$. It readily follows that Hyperlat T = Lat T if and only if for every invertible operator S commuting with T satisfying $\operatorname{Lat} S = \operatorname{Lat} S^{-1}$ the mapping $M \to SM$ $(M \in \operatorname{Lat} T)$ is the identity automorphism.

Proposition 2.2. *If* Lat T *is distributive and* S *is an invertible operator commuting with* T, *then* $M \subseteq SM \vee S^{-1}M$ $(M \in \operatorname{Lat} T)$.

Proof. Choose λ with $0 < \lambda < 1/\|S\|$. The operator $C = 1 + \lambda S$ is invertible and commutes with T. Let $M \in \operatorname{Lat} T$. It is readily verified that $CM \cap SM = C(M \cap SM)$ and $CM \cap M = C(M \cap S^{-1}M)$. Since $CM \subseteq SM \vee M$, by distributivity we have

$$\begin{aligned} CM &= (CM \cap SM) \vee (CM \cap M) = C(M \cap SM) \vee C(M \cap S^{-1}M) \\ &= C(M \cap [SM \vee S^{-1}M]) \end{aligned}$$

and the result follows.

Theorem 2.3. *If* Lat T *is σ-infinitely meet distributive* Hyperlat T = Lat T.

Proof. By our earlier remarks it suffices to show that if S is an invertible operator commuting with T and satisfying $\operatorname{Lat} S = \operatorname{Lat} S^{-1}$ then the automorphism $M \to SM$ of Lat T is the identity automorphism. Since Lat T is distributive, this automorphism satisfies condition (I) of Proposition 2.1 by Proposition 2.2. Since $\operatorname{Lat} S = \operatorname{Lat} S^{-1}$, condition (II) is also satisfied. The result now follows from Proposition 2.1.

Corollary 2.3.1. Hyperlat T = Lat T if Lat T is any one of the following:
(i) *commutative*;
(ii) *isomorphic to the direct product of complete chains*;
(iii) *totally ordered*.

PROOF. It is clear that in cases (ii) and (iii) Lat T is σ-infinitely meet distributive. Suppose that Lat T is commutative. Then it is also distributive. This follows from the fact that if P and Q are commuting projections then PQ is the projection onto the intersection of the ranges of P and Q and $P + Q - PQ$ is the projection onto the closed linear span of the ranges of P and Q. Thus if $K, L, M \in$ Lat T then

$$\begin{aligned} P_{K\cap(L\vee M)} &= P_K P_{L\vee M} = P_K(P_L + P_M - P_L P_M) \\ &= P_{K\cap L} + P_{K\cap M} - P_{K\cap L}P_{K\cap M} = P_{(K\cap L)\vee(K\cap M)}. \end{aligned}$$

Now let

$$K, L_n \ (n \geqslant 1) \in \text{Lat } T.$$

Then $P_{\cap\{L_n:\ n\geqslant 1\}}$, respectively $P_{\cap\{K\vee L_n:\ n\geqslant 1\}}$, is the strong limit of the sequence $\{P_{\cap\{L_n:1\leqslant n\leqslant k\}}: k \geqslant 1\}$, respectively $\{P_{\cap\{K\vee L_n:1\leqslant n\leqslant k\}}: k \geqslant 1\}$. But

$$\begin{aligned} P_{\cap\{K\vee L_n:1<n<k\}} &= P_{K\vee\cap\{L_n:\ 1<n<k\}} \\ &= P_K + P_{\cap\{L_n:1<n<k\}} - P_K P_{\cap\{L_n:1<n<k\}}. \end{aligned}$$

Taking strong limits gives

$$P_{\cap\{K\vee L_n:\ n>1\}} = P_K + P_{\cap\{L_n:n>1\}} - P_K P_{\cap\{L_n:n>1\}} = P_{K\vee\cap\{L_n:n>1\}}.$$

Hence Lat T is σ-infinitely meet distributive. The result follows by applying Theorem 2.3.

The results (i) and (iii) above are not new. The former was proved in [2] and the latter in [3].

3. **Concluding remarks.** It is a simple exercise to show for any linear transformation T on a finite-dimensional complex linear space that Hyperlat T is distributive and finite, therefore σ-infinitely meet distributive. It seems an interesting question whether Hyperlat T is always σ-infinitely meet distributive or even whether the converse of Theorem 2.3 holds. For a normal operator T, Hyperlat T consists of the ranges of the spectral projections for T [2] and so is both commutative and a Boolean algebra and so is certainly σ-infinitely meet distributive.

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