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A GEOMETRIC EQUIVALENT OF THE INVARIANT SUBSPACE PROBLEM

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ABSTRACT. It is shown that every operator has an invariant subspace if and only if every pair of idempotents has a common invariant subspace.

An *idempotent* on a Hilbert space is a bounded linear operator whose square is itself. Obviously each idempotent has nontrivial invariant subspaces. On the other hand, a well-known result of Chandler Davis [1] implies that there exist three selfadjoint idempotents which have no common nontrivial invariant subspaces. We show that each pair of idempotents has a common nontrivial invariant subspace if and only if each bounded linear operator has a nontrivial invariant subspace.

The two lemmas below follow from direct computations; the second is due to Davis [1].

LEMMA 1. *If A is an operator on $\mathcal{H}$, then $\begin{pmatrix} A & A \\ 1-A & 1-A \end{pmatrix}$ is an idempotent on $\mathcal{H} \oplus \mathcal{H}$.*

LEMMA 2. *If P and Q are idempotents, then $[PQP + (1-P)(1-Q)(1-P)]$ commutes with P and with Q.*

Note that Lemma 1 implies that every operator is the compression of an idempotent. This might be compared with the fact [2] that every selfadjoint operator between 0 and 1 is the compression of a selfadjoint idempotent.

For the theorem below we assume the space is separable and infinite dimensional; it is easy to see that the theorem holds on nonseparable spaces and on finite-dimensional spaces of dimension at least three, in which cases all the equivalent statements are true.

THEOREM. *For operators on a Hilbert space the following are equivalent*:

(i) *every bounded operator has a nontrivial invariant subspace*,

(ii) *each pair of idempotents has a common nontrivial invariant subspace*,

(iii) *each pair of similar idempotents has a common nontrivial invariant subspace*,

(iv) *if P is any idempotent and S is any invertible operator, then there exists a nontrivial invariant subspace $\mathfrak{M}$ of P such that $S\mathfrak{M}$ is invariant under P*,

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(v) *if* $\mathcal{H} = \mathcal{K} + \mathcal{L} = \mathcal{M} + \mathcal{N}$ *are any direct-sum decompositions of* $\mathcal{H}$ (*with* $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ *and* $\mathcal{N}$ *closed*), *then there exist subspaces* $\mathcal{K}_0$, $\mathcal{L}_0$, $\mathcal{M}_0$ *and* $\mathcal{N}_0$ *of* $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ *and* $\mathcal{N}$ *respectively such that* $\mathcal{K}_0 + \mathcal{L}_0 = \mathcal{M}_0 + \mathcal{N}_0$ *and is not* $\{0\}$ *or* $\mathcal{H}$.

PROOF. Assume (i) and let P and Q be idempotents. If P has rank 1 let $\mathcal{S} = P\mathcal{H}$. If P has rank greater than 1, then the restriction of PQP to the range of P has a nontrivial invariant subspace, (by (i)). In this case let $\mathcal{S}$ be such an invariant subspace. In either case set $\mathcal{T} = \mathcal{S} + \overline{(1-P)Q\mathcal{S}}$; we claim that $\mathcal{T}$ is a common invariant subspace of P and Q. Now $P\mathcal{T} = \mathcal{S}$, so $\mathcal{T}$ is invariant under P. Also $PQ\mathcal{S} \subset \mathcal{S}$, and $(1-P)Q\mathcal{S} \subset \mathcal{T}$, so $Q\mathcal{S} \subset \mathcal{T}$ and it need only be shown that $Q(1-P)Q\mathcal{S} \subset \mathcal{T}$. But $Q(1-P)Q\mathcal{S} \subset Q\mathcal{S} + QPQ\mathcal{S} \subset Q\mathcal{S} + Q\mathcal{S} \subset \mathcal{T}$. Therefore (i) implies (ii).

To show that (ii) implies (i) let A be any given operator. If A has compression spectrum we are done, so assume that the range of $A - \lambda$ is dense for all complex λ. Let

$$P = \begin{pmatrix} A & A \\ 1-A & 1-A \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

P is an idempotent by Lemma 1. The algebra generated by 1, P and Q clearly contains all operators of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with each A_i a polynomial in A. It also contains the operators $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1-A & 0 \end{pmatrix}$. Let $\mathcal{S}$ be a common invariant subspace of P and Q; then $\mathcal{S} = Q\mathcal{S} \oplus (1-Q)\mathcal{S}$. If $Q\mathcal{S} = \mathcal{H}$, then $\mathcal{S}$ contains all vectors of the form $x \oplus (1-A)y$ for $x, y \in \mathcal{H}$, so $\mathcal{S} = \mathcal{H} \oplus \mathcal{H}$ (we are assuming that A has no compression spectrum). Similarly, if $(1-Q)\mathcal{S} = \mathcal{H}$ then $\mathcal{S}$ contains all vectors of the form $Ax \oplus y$ for $x, y \in \mathcal{H}$, and $\mathcal{S} = \mathcal{H} \oplus \mathcal{H}$. Thus, if $\mathcal{S}$ is nontrivial it follows that both $Q\mathcal{S}$ and $(1-Q)\mathcal{S}$ are proper subspaces of $\mathcal{H}$. Now $Q\mathcal{S}$ and $(1-Q)\mathcal{S}$ are each invariant under A; if $\mathcal{S}$ is nontrivial then at least one of these subspaces is nontrivial. Therefore (ii) implies (i).

On a separable space two idempotents are similar unless at least one of them has finite-dimensional range or nullspace. Suppose P or $(1-P)$ has finite rank. Then $PQP + (1-P)(1-Q)(1-P)$ has point spectrum. If $PQP + (1-P)(1-Q)(1-P)$ is a multiple of the identity, then each vector in the range of $1-P$ generates a common invariant subspace of dimension less than 3. (If $x = (1-P)x$, then the linear span of PQx and x is invariant under Q, since PQP is a multiple of P and $(1-P)Q(1-P)$ is a multiple of $(1-P)$.) Lemma 2 gives the equivalence of (ii) and (iii).

If $Q = S^{-1}PS$ and $\mathcal{M}$ is invariant under P and Q, then $Q\mathcal{M} \subset \mathcal{M}$ implies $PS\mathcal{M} \subset S\mathcal{M}$, so (iii) and (iv) are equivalent. The equivalence of (ii) and (v) is obvious.

Note that (i) and (ii) are equivalent for Banach spaces too (although the space must be changed). Note also that Lemma 1 implies that the transitive and reductive algebra problems (see [3]) are equivalent to the cases in which the algebras are generated by idempotents.

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