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MINIMAL SURFACES IN TORI BY WEYL GROUPS

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ABSTRACT. We report a method of constructing compact minimal surfaces immersed in flat Riemannian tori of arbitrary dimension $n \geqslant 3$ by the use of the Weyl groups of compact simple Lie groups, inspired by H. A. Schwarz [5].

1. **Minimal surfaces**. The construction is based on two known theorems on minimal surfaces immersed in the euclidean space $\mathbf{R}^n$ (see Lawson [2] or Osserman [4]). Given a Jordan curve P in $\mathbf{R}^n$, there exists a unique minimal surface $M(P)$ embedded in $\mathbf{R}^n$ and bounded by P if P satisfies the following condition (Rado's theorem):

(R) *There exists an affine projection of* $\mathbf{R}^n$ *onto some* 2*-plane which sends* P *bijectively onto the boundary of a convex domain* D *in the plane.*

Next we assume that P is a skew-polygon satisfying (R); thus P consists of several edges. Since each edge, e, is a segment of a line, there corresponds the symmetry $s = s(e)$ in the line. Obviously s carries $M(P)$ onto a minimal surface $sM(P)$, which is the same as $M(sP)$ by the uniqueness in Rado's theorem. Moreover the Schwarz reflection principle guarantees that the union $M(P) \cup sM(P)$ is also an immersed minimal surface; that is, the union has no singularities along the edge e except possibly at the end points of e. We repeat this process endlessly to obtain a complete surface; more precisely, letting S denote the transformation group of $\mathbf{R}^n$ generated by all the symmetries in the edges of P, we consider the orbit $S(M(P)) = \bigcup_{s \in S} s(M(P))$, which is an immersed minimal surface without singularities except possibly at the vertices of the 1-complex $S(P)$. To eliminate this possibility, we have to choose a more special polygon.

2. **Use of Weyl groups**. We shall construct an example of a nice P, postponing a more systematic account which will require some knowledge of Weyl groups as is explained in [1]. Let $(e_i)_{1 \leqslant i \leqslant n}$ be an orthonormal basis for $\mathbf{R}^n$. Consider the set R consisting of the vectors $\pm e_i$, $1 \leqslant i \leqslant n$, $\pm e_i \pm e_j$, $1 \leqslant i < j \leqslant n$. Call them *the roots* (of $SO(2n+1)$). Let W, the Weyl group of R, be the transformation group of $\mathbf{R}^n$ generated by the reflections in the hyperplanes through the origin 0 normal to the roots. This amounts, in this case of $SO(2n+1)$, to the same as the group generated by the symmetries in

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the lines through 0 spanned by the roots. The group W is finite and leaves invariant the lattice group L spanned by the roots over the integers; L is the translation subgroup $\mathbf{Z}^n$ of $\mathbf{R}^n$ for our R. We wish to construct a polygon P which satisfies these conditions: (1) *P satisfies Rado's condition*; (2) *the vertices of P, viewed as position vectors, are contained in L; and* (3) *the edges of P are parallel to roots*. Let P be the skew-polygon having the $n+1$ vertices 0, $e_1 - e_2, \dots, e_1 - e_k, \dots, e_1 - e_n$ and e_1 in this order. (For another example one may replace the last vector e_1 by $-e_n$.) The edges of P are therefore parallel to the roots $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n$ and $-e_1$. Clearly P satisfies (2) and (3). That P satisfies (1) will be easily seen if one notes that condition (R) is affine in nature and that the convex hull of P is the affine n-simplex (of which P is the unique 1-subcomplex containing all the vertices, to within affine isomorphism). Observe that L is a normal subgroup of S and the linear part S/L is the Weyl group W; in particular W is conjugate to the isotropy subgroup of S at each vertex of P.

We now construct an immersion f': $M' \to \mathbf{R}^n$ of a smooth surface M', whose image is $SM(P)$ and which induces an immersion f: $M \to T = \mathbf{R}^n/L$ of a compact surface M. First we assign a subgroup $S_{(x)}$ of S to each point x of $M(P)$. According as x is an interior point of $M(P)$, a point on an edge e of P different from the end points or a vertex of P, we denote by $S_{(x)}$ the identity $\{1\}$, the group $\{1, s(e)\}$ or the group generated by $\{s(a), s(b)\}$, $a \cap b = x$, respectively. Let M' be the quotient space of $M(P) \times S$ by the equivalence relation identifying (x, g) with (y, gh) under the condition $gx = ghy$ for some $h \in S_{(x)}$. The immersion f' is induced by the map: $(g, x) \to gx$. That M' has no singularities at the vertices x may be seen based on the fact that $S_{(x)}$ acts on the plane $p(x)$ spanned by the incident edges a, b and P satisfies the condition (4): *the cone* (*or the sector*) *determined by a, b in $p(x)$ is a fundamental domain of $S_{(x)}$ acting on $p(x)$*. The left action of S on itself induces an action of S on M' in the obvious manner, and this gives rise to the compact surface $M = M'/L$ as well as the W-equivariant immersion f: $M \to T$ with $f(M) = S(M(P))/L$. In passing let us note these facts: f is a minimal immersion w.r.t. the "standard" flat metric on T; W acts effectively on M and has order $n!2^n$.

3. THEOREM. *For each integer $n \geqslant 3$ and each irreducible* (*and reduced*) *root system R of rank n, there exists a minimal isometric immersion f of a compact 2-dimensional Riemannian manifold M into a flat Riemannian torus T of dimension n for which* (i) *the image $f(M)$ lies in no subtorus of T,* (ii) *the Gauss curvature* (= *the Ricci form*) *of M is negative except possibly at a finite number of points, and* (iii) *the isometry group $I(M)$ has order $\geqslant n!$, or more strongly there is an action of the Weyl group of R on M as a subgroup of $I(M)$ which makes f equivariant.*

REMARKS ON THE STATEMENT OF THE THEOREM. The metric of T is not the standard one in general. By (ii) the isometry group is necessarily finite. If M is orientable, then M becomes a Riemann surface in the well-known way and M admits a holomorphic transformation group which is isomorphic with a subgroup of W of index 2.

OUTLINE OF THE PROOF. Fix a Weyl chamber C of R. Let $B(C)$ be a basis associated with C, also called a simple root system. Let ρ_n be the highest root. Then we can choose members $\alpha_2, \ldots, \alpha_n$ of $B(C)$ such that the differences $\rho_k = \rho_{k+1} - \alpha_{k+1}$, $1 \leqslant k < n$, defined inductively are all roots. Let P be the $(n+1)$-gon having the vertices $0, \rho_1, \rho_2, \ldots, \rho_n$ in this order. Clearly P satisfies conditions (1), (2) and (3). It follows that $L' = S \cap L$ is a normal subgroup of S such that $\mathbf{R}^n/L'$ is a torus. We have $L' = L$ by a suitable choice of α's (which is necessary for the R of type E). Since W may not contain the inversion -1, the linear part S/L may not be isomorphic with W, but their extensions by $\{\pm 1\}$ are identical. The rest goes through as before to give a minimal W-equivariant immersion $f\colon M \to T$ whose image is the compact set $S(M(P))/L$. Induce a Riemannian metric on M from T by f. Then we have condition (iii) of the theorem. Also we have (i) since $B(C)$ is linearly independent and so is $(\rho_k)_{1 \leqslant k \leqslant n}$. Finally one easily infers (ii) from (i) by our previous result [3]: to wit, if a compact Riemannian manifold M (of any dimension) is minimally and isometrically immersed in a flat Riemannian torus T, then, by passing to some finite covering of M, we have that M is the de Rham product of a flat torus T_1 and a subspace M_1 whose Ricci form is negative definite almost everywhere and f accordingly decomposes into the product of a totally geodesic immersion of T_1 and a minimal isometric immersion of M both into flat tori.

REMARKS. P is not unique for a given R by the method above; even the change of the order of α's can give a different M. In the case of $n = 3$, the method exhausts the surfaces in [5] and [6] (with some modifications; e.g. the surface IV of [6] has branching points of order 1). There may be a more general method. Given any root which is the sum of n positive roots, we can similarly construct a P which satisfies (1), (2) and (3), but not (4) in general. The conclusion (iii) may not obtain either.

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