PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

ON POSITIVELY TURNING IMMERSIONS

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ABSTRACT. Let $\gamma: S^1 \to \mathbf{C}$ be a C^2 immersion of the circle. Let k be the number of zeros of γ and suppose $d \arg \gamma(e^{i\theta})/d\theta > 0$ for $\gamma(e^{i\theta}) \neq 0$; then $\operatorname{twn} \gamma = k/2 + (2\pi)^{-1} \int_A d \arg \gamma$ where $\operatorname{twn} \gamma$ is the tangent winding number, and $A = S^1 - \gamma^{-1}(0)$. This generalizes the theorem of Cohn that if p is a self-inversive polynomial, the number of zeros of p' in $|z| > 1$ is the same as the number of zeros of p in $|z| > 1$. For $k = 0$, this is a topological generalization of Lucas' theorem. We show how $(2\pi)^{-1} \int_A d \arg \gamma$ represents a generalization of the notion of the winding number of γ about 0.

Introduction. The classical theorem of Lucas says that if p is a polynomial and all the zeros of p are in $|z| < 1$, then all the zeros of p' are in $|z| < 1$. We may prove the theorem topologically as follows: Since all the zeros of p are in $|z| < 1$, we readily see that $d \arg p(e^{i\theta})/d\theta > 0$ for all θ. Thus $p(e^{i\theta})$ is a regular closed curve such that the vector from the origin to a point on the curve always turns in a positive (counterclockwise) direction. For such a curve it is seen that the tangent winding number is equal to the winding number about zero. The tangent winding number is just 1 + (number of zeros of p' in $|z| < 1$) and this proves Lucas' theorem. In this paper, we show that a theorem of Cohn on self-inversive polynomials can be generalized in a similar way to a theorem relating the tangent winding number and the winding number about zero of a regular curve γ on which $\arg \gamma$ is increasing except where γ is zero. The theorem that we prove has a topological generalization of Lucas' theorem as a special case. In the course of the proof, we offer a generalization of the notion of winding number $\omega(\gamma, a)$ to cover the case when a is on the image of γ.

We remark that the proof given here of Cohn's theorem is similar to a proof in Bonsall and Marden [1] that was suggested by J. L. Walsh. Other information on positively turning curves may be found in Pólya and Szegö [4, part three, problems 103–111].

1. **A generalization of Cohn's theorem.** Let p be a polynomial of degree n. Say p is self-inversive if the zeros of p are symmetric in the circle $|z| = 1$, i.e., $p(z) = 0$ iff $p(1/\bar{z}) = 0$. Now p is self-inversive iff $z^n \overline{p(1/\bar{z})} = cp(z)$ where $|c| = 1$. Let $c = e^{2i\phi}$. Then we have

$$p(e^{i\theta})e^{i(\phi - n\theta/2)} = \overline{p(e^{i\theta})}e^{-i(\phi - n\theta/2)}$$

Received by the editors November 3, 1975.

AMS (MOS) subject classifications (1970). Primary 30A90, 30A06; Secondary 57D40.

Key words and phrases. Tangent winding number, Cohn's theorem, Lucas' theorem, self-inverse polynomials, immersions.

so that we may write $p(e^{i\theta}) = R(\theta)e^{-i(\phi - n\theta/2)}$ where $R(\theta)$ is real. Therefore for $p(e^{i\theta}) \neq 0$,

$$d\arg p(e^{i\theta})/d\theta = \operatorname{Im}\,(ie^{i\theta}p'(e^{i\theta})/p(e^{i\theta})) = n/2.$$

Thus these polynomials have the interesting property that the ray from 0 to $p(e^{i\theta})$ turns at a constant positive angular velocity except where $p(e^{i\theta}) = 0$. We note also that $p'(z) \neq 0$ except at the multiple zeros of p. We have the following theorem of Cohn on self-inversive polynomials (see Cohn [2] and Marden [3]): If p is self-inversive then the number of zeros of p' in $|z| > 1$ is equal to the number of zeros of p in $|z| > 1$. In what follows we will prove a generalization of Cohn's theorem to closed curves in $\mathbf{C}$ with positively changing argument.

Let $S^1 = \{z \,|\, |z| = 1\} \subseteq \mathbf{C}$, considered as a 1-manifold with coordinate systems $\theta \to e^{i\theta}$. We will consider a closed curve as a C^2 map $\gamma\colon S^1 \to \mathbf{C}$. We will suppose also that $d\gamma(z)/d\theta \neq 0$ for $z \in S^1$, so that γ is an immersion. Let $[\gamma]$ denote the image of γ. If $0 \notin [\gamma]$ then the winding number of γ about 0, $\omega(\gamma, 0)$, is defined to be $(2\pi)^{-1}\int_{S^1} d\arg\gamma$. We may drop the condition $0 \notin [\gamma]$ and extend the definition, as the following discussion shows. Suppose $\gamma(e^{i\phi}) = 0$. Then

$$\begin{aligned}\lim_{\theta\to\phi}\frac{d\arg\gamma(e^{i\theta})}{d\theta} &= \lim_{\theta\to\phi}\operatorname{Im}\left(\frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})}\right)\\ &= \lim_{\theta\to\phi}\operatorname{Im}\left(\frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})} - \frac{1}{(\theta-\phi)}\right)\\ &= \operatorname{Im}\left(\frac{d^2\gamma(e^{i\phi})/d\theta^2}{2d\gamma(e^{i\phi})/d\theta}\right),\end{aligned}$$

Thus we may extend

$$\operatorname{Im}\left(\frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})}\right)d\theta = d\arg\gamma(e^{i\theta})$$

to a continuous differential on S^1 and define the integral over S^1 as $\omega(\gamma, 0)$. We remark here for future reference that

$$\lim_{\theta\to\phi^+}\operatorname{Re}\left(\frac{d\gamma(e^{i\theta})/d\theta}{\gamma(e^{i\theta})}\right) = \lim_{\theta\to\phi^+}\frac{1}{\theta-\phi}\operatorname{Re}\left(\frac{d\gamma(e^{i\theta})}{d\theta}\,\frac{(\theta-\phi)}{\gamma(e^{i\theta})}\right) = +\infty$$

and similarly as $\theta \to \phi^-$, the limit is $-\infty$.

Now let $\beta = d\gamma/d\theta$. We define the tangent winding number of γ, $\operatorname{twn}\gamma$, to be $\omega(\beta, 0)$. We have the following theorem.

THEOREM 1. *Suppose $\gamma\colon S^1 \to \mathbf{C}$ is a C^2 immersion. Let k be the number of zeros of γ, and suppose $d\arg\gamma(z)/d\theta > 0$ for $\gamma(z) \neq 0$. Then* $\operatorname{twn}\gamma = k/2 + \omega(\gamma, 0)$.

PROOF. First suppose $\gamma^{-1}(0)$ is not the null set, and let $\gamma^{-1}(0) = \{z_1, \ldots,$

$z_k\}$. Let A_j be the arc of S^1 from z_j to z_{j+1} for $j = 1, \ldots, k-1$ and A_k be the open arc from z_k to z_1. Let $\beta = d\gamma/d\theta$; then $d\arg\gamma/d\theta = \operatorname{Im}(\beta/\gamma)$ on $S^1 - \gamma^{-1}(0)$. Let $z_j = \exp(i\theta_j)$ for $j = 1, \ldots, k$. By our previous remark,

$$\lim_{\theta\to\theta_j^+} \operatorname{Re}\,(\beta(e^{i\theta})/\gamma(e^{i\theta})) = +\infty$$

and as $\theta \to \theta_j^-$, the limit is $-\infty$. Since by hypothesis, $\operatorname{Im}(\beta/\gamma) > 0$ on $A_j, j = 1, \ldots, k$, we conclude that $\int_{A_j} d\arg(\beta/\gamma) = \pi, j = 1, \ldots, k$. We also have $d\arg\beta = d\arg(\beta/\gamma) + d\arg\gamma$. Integrating over $S^1 - \gamma^{-1}(0)$ and dividing by 2π, we get the result.

Now if $\gamma^{-1}(0)$ is empty, then since $\operatorname{Im}(\beta/\gamma) > 0$ on S^1, we have $\int_{S^1} d\arg(\beta/\gamma) = 0$ and $\operatorname{twn}\beta = \omega(\gamma, 0)$ and the theorem holds with $k = 0$. This completes the proof of the theorem.

We now show how Theorem 1 proves the theorem of Cohn. We remark first that it is enough to prove the theorem when none of the zeros of p on $|z| = 1$ are multiple because there are self-inversive polynomials q_ε near p with distinct roots on $|z| = 1$, and such that $\lim_{\varepsilon\to 0} q_\varepsilon = p$. Now q'_ε has no zeros on $|z| = 1$ so the number of zeros of q_ε and q'_ε in $|z| > 1$ remains the same in taking the limit. Now suppose p is self-inversive of degree n, where none of the zeros of p on $|z| = 1$ are multiple. i.e., $p'(z) \neq 0$ on $|z| = 1$. Let l be the number of zeros of p' in $|z| > 1$ and let m be the number of zeros of p in $|z| > 1$. Then the number of zeros of p on $|z| = 1$ is $n - 2m$ and the number of zeros of p in $|z| < 1$ is $n - k - 1$. Since $d\arg p(e^{i\theta})/d\theta = n/2$, the theorem and the argument principle give $n - l = (n - 2m)/2 + n/2$; hence $l = m$.

The theorem for $k = 0$ is a generalization of Lucas' theorem for polynomials.

2. **The generalized winding number.** We will investigate further the meaning of the generalized winding number defined in §1. We first give some definitions. Let $I = [0, 1]$ and let $I \times S^1$ have coordinate systems given by $(t, \phi) \to (t, e^{i\phi})$. Let F: $I \times S^1 \to \mathbf{C}$ be C^2. Say F is a positive monotopy if the Jacobian is positive on $I \times S^1$ in these coordinate systems.

Now let γ be a C^2 immersion and suppose F is a positive monotopy such that $F(\frac{1}{2}, z) = \gamma(z)$ for $z \in S^1$ and such that the only zeros of F are on $\frac{1}{2} \times S^1$. Let $F(0, z) = \alpha(z)$ and $F(1, z) = \beta(z)$. Then we define $\omega^+(\gamma, 0) = \omega(\beta, 0)$ and $\omega^-(\alpha, 0) = (\alpha, 0)$. Such a monotopy may be constructed by setting

$$F(t, z) = \gamma(z) + (t - \tfrac{1}{2})ci\frac{d\gamma}{d\theta}(z)$$

for suitably small $c > 0$. To show that ω^+ and ω^- are defined independently of F and to show how they relate to the generalized winding number we prove the next theorem.

THEOREM 2. *Let* γ: $S^1 \to \mathbf{C}$ *be a* C^2 *immersion and let* k *be the number of zeros of* γ. *Then* $\omega(\gamma, 0) = (\omega^+(\gamma, 0) + \omega^-(\gamma, 0))/2$; $k = \omega^+(\gamma, 0) - \omega^-(\gamma, 0)$.

PROOF. Write $f(t, x) = F(t, e^{2\pi ix})$. Let $X = I \times I$. Then f: $X \to \mathbf{C}$. Let $X^+ = \{(t, x) | t > \frac{1}{2}\}$, $X^- = \{(t, x) | t < \frac{1}{2}\}$. Let $\{P_1, \ldots, P_k\} = F^{-1}(0)$. By assumption $F^{-1}(0) \in \frac{1}{2} \times I$. Let

$$D_j = \{P \in I \times I | \text{dist}\,(P, P_k) \leqslant \varepsilon\}, \qquad j = 1, \ldots, k,$$

where ε is chosen so that $D_j \subseteq X$ for $j = 1, \ldots, k$. Let ∂D_j be the positively oriented boundary of D_j. Since f has positive Jacobian, we may choose ε so that $\int_{\partial D_j} df/f = 2\pi, j = 1, \ldots, k$. Let $\partial D_j^+ = \partial D_j \cap X^+$ and $\partial D_j^- = \partial D_j \cap X^-$, both considered as paths for $j = 1, \ldots, k$. Let ∂D^+ by the cycle $\sum_{j=1}^n \partial D_j^+$ and ∂D^- be the cycle $\sum_{j=1}^n \partial D_j^-$. Let σ_ε be the cycle consisting of the line $t = \frac{1}{2}$ minus $\bigcup_{j=1}^n D_j$, traversed in the direction of increasing x. Since df/f is a closed differential in $X - \{P_1, \ldots, P_k\}$ we have

$$\omega(\alpha, 0) = \frac{1}{2\pi} \int_{\sigma_\varepsilon} \frac{df}{f} + \frac{1}{2\pi} \int_{\partial D^-} \frac{df}{f} \tag{1}$$

and

$$\omega(\beta, 0) = \frac{1}{2\pi} \int_{\sigma_\varepsilon} \frac{df}{f} - \frac{1}{2\pi} \int_{\partial D^+} \frac{df}{f}. \tag{2}$$

We also verify that

$$\lim_{\varepsilon \to 0} \int_{\partial D_k^+} \frac{df}{f} = \pi; \qquad \lim_{\varepsilon \to 0} \int_{\partial D_k^-} \frac{df}{f} = \pi.$$

Now adding and subtracting (1) and (2) and taking the limit gives the theorem.

Thus if we "expand" or "contract" γ slightly and take the average winding number about 0, we get $\omega(\gamma, 0)$.

We remark that the conclusion to Theorem 1 may now be stated as twn $\gamma = \omega^+(\gamma, 0)$.

References

1. F. F. Bonsall and M. Marden, *Zeros of self-inversive polynomials*, Proc. Amer. Math. Soc. **3** (1952), 471–475. MR **13**, 938.

2. A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Z. **14** (1922), 110–148.

3. M. Marden, *Geometry of polynomials*, 2nd ed., Math. Surveys, no. 3, Amer. Math. Soc., Providence, R.I., 1966. MR **37** #1562.

4. G. Polya and G. Szegö, *Problems and theorems in analysis*, Vol. I, Springer-Verlag, Berlin and New York, 1972. MR **49** #8782.

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