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PERIODIC POINTS OF CONTINUOUS MAPS AND LINDEMANN'S INDEPENDENCE THEOREM FOR EXPONENTIALS

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ABSTRACT. We give a simple proof of the sufficiency of known conditions for the existence of periodic points of a continuous map, using a classical theorem of Lindemann on transcendental numbers.

The purpose of this note is to give a very simple proof of a generalization of the following well-known theorem of F. B. Fuller:

THEOREM 1 (FULLER). *If ϕ is a self-homeomorphism of a finite complex X, and $\chi(X) \neq 0$, then ϕ has a periodic point.*

We consider a space X for which the Lefschetz Fixed Point Theorem is valid, and a map $\phi: X \to X$. Let $\phi_{*q}: H_q(X) \to H_q(X)$ $(q \geqslant 0)$ denote the induced maps of the rational homology of X; set $\lambda_0 = 0$, and let $\lambda_1, \ldots, \lambda_n$ be the distinct nonzero eigenvalues of the ϕ_{*q}; let $m_q(\lambda_j)$ be the multiplicity of λ_j in ϕ_{*q} $(j = 0, \ldots, n)$. We then have:

THEOREM 2. *If ϕ is a self-map of a Lefschetz space X and ϕ has no periodic points, then*

$$\sum_q (-1)^q m_q(\lambda_0) = \chi(X) \quad \textit{and} \quad \sum_q (-1)^q m_q(\lambda_j) = 0 \qquad (j = 1, \ldots, n).$$

Theorem 1 has been considerably extended by Fuller [2] and others: the survey [1] of Fadell contains an elegant exposition of these results. Theorem 2 also follows from these extensions, but our direct proof is somewhat simpler.

PROOF OF THEOREM 2. Let m be the maximum dimension for which $H_q(X) \neq 0$. For $q = 0, \ldots, m$, let A_q be a matrix representing ϕ_{*q} with respect to some fixed basis for $H_q(X)$. Define

$$B_k = \operatorname{diag}[A_0^k, -A_1^k, \ldots, (-1)^m A_m^k],$$

for $k \geqslant 0$, and set

$$E = \sum_{k=0}^{\infty} \frac{1}{k!} B_k. \tag{1}$$

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Note that $\mathrm{Tr}(B_0) = \chi(X)$, and, for $k \geqslant 1$, $\mathrm{Tr}(B_k) = \Lambda(\phi^k)$, the Lefschetz number of ϕ^k. If ϕ has no periodic points then, by the Lefschetz Fixed Point Theorem, $\mathrm{Tr}(B_k) = 0$ for $k \geqslant 1$. We then have,equating traces in (1):

$$\sum_{j=0}^{n} \sum_{q=0}^{m} (-1)^q m_q(\lambda_j) e^{\lambda_j} = \chi(X) \cdot e^{\lambda_0}. \tag{2}$$

But the A_q are rational matrices, so the eigenvalues $\lambda_0, \ldots, \lambda_n$ are distinct algebraic numbers, and we can apply the theorem of Lindemann ([3, p. 117], e.g.) that asserts the linear independence over the field of algebraic numbers of $e^{\lambda_0}, e^{\lambda_1}, \ldots, e^{\lambda_n}$, to conclude that each of the coefficients in equation (2) vanishes. This is just the conclusion of Theorem 2.

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