

PERIODIC POINTS OF CONTINUOUS MAPS AND LINDEMANN'S INDEPENDENCE THEOREM FOR EXPONENTIALS

DEAN A. NEUMANN

ABSTRACT. We give a simple proof of the sufficiency of known conditions for the existence of periodic points of a continuous map, using a classical theorem of Lindemann on transcendental numbers.

The purpose of this note is to give a very simple proof of a generalization of the following well-known theorem of F. B. Fuller:

THEOREM 1 (FULLER). *If ϕ is a self-homeomorphism of a finite complex X , and $\chi(X) \neq 0$, then ϕ has a periodic point.*

We consider a space X for which the Lefschetz Fixed Point Theorem is valid, and a map $\phi: X \rightarrow X$. Let $\phi_{*q}: H_q(X) \rightarrow H_q(X)$ ($q \geq 0$) denote the induced maps of the rational homology of X ; set $\lambda_0 = 0$, and let $\lambda_1, \dots, \lambda_n$ be the distinct nonzero eigenvalues of the ϕ_{*q} ; let $m_q(\lambda_j)$ be the multiplicity of λ_j in ϕ_{*q} ($j = 0, \dots, n$). We then have:

THEOREM 2. *If ϕ is a self-map of a Lefschetz space X and ϕ has no periodic points, then*

$$\sum_q (-1)^q m_q(\lambda_0) = \chi(X) \quad \text{and} \quad \sum_q (-1)^q m_q(\lambda_j) = 0 \quad (j = 1, \dots, n).$$

Theorem 1 has been considerably extended by Fuller [2] and others: the survey [1] of Fadell contains an elegant exposition of these results. Theorem 2 also follows from these extensions, but our direct proof is somewhat simpler.

PROOF OF THEOREM 2. Let m be the maximum dimension for which $H_q(X) \neq 0$. For $q = 0, \dots, m$, let A_q be a matrix representing ϕ_{*q} with respect to some fixed basis for $H_q(X)$. Define

$$B_k = \text{diag}[A_0^k, -A_1^k, \dots, (-1)^m A_m^k],$$

for $k \geq 0$, and set

$$(1) \quad E = \sum_{k=0}^{\infty} \frac{1}{k!} B_k.$$

Received by the editors May 24, 1976.

AMS (MOS) subject classifications (1970). Primary 55C20; Secondary 54H25.

Key words and phrases. Lefschetz number, fixed points for iterates.

Copyright © 1977, American Mathematical Society

Note that $\text{Tr}(B_0) = \chi(X)$, and, for $k \geq 1$, $\text{Tr}(B_k) = \Lambda(\phi^k)$, the Lefschetz number of ϕ^k . If ϕ has no periodic points then, by the Lefschetz Fixed Point Theorem, $\text{Tr}(B_k) = 0$ for $k \geq 1$. We then have, equating traces in (1):

$$(2) \quad \sum_{j=0}^n \sum_{q=0}^m (-1)^q m_q(\lambda_j) e^{\lambda_j} = \chi(X) \cdot e^{\lambda_0}.$$

But the A_q are rational matrices, so the eigenvalues $\lambda_0, \dots, \lambda_n$ are distinct algebraic numbers, and we can apply the theorem of Lindemann ([3, p. 117], e.g.) that asserts the linear independence over the field of algebraic numbers of $e^{\lambda_0}, e^{\lambda_1}, \dots, e^{\lambda_n}$, to conclude that each of the coefficients in equation (2) vanishes. This is just the conclusion of Theorem 2.

BIBLIOGRAPHY

1. E. R. Fadell, *Recent results in the fixed point theory of continuous maps*, Bull. Amer. Math. Soc. **76** (1970), 10–29. MR **42** #6816.
2. F. B. Fuller, *Periodic trajectories of a one-parameter semigroup*, Bull. Amer. Math. Soc. **69** (1963), 409–410. MR **26** #4002.
3. Ivan Niven, *Irrational numbers*, Math. Assoc. Amer. Carus Monograph No. 11, Wiley, New York, 1956. MR **18**, 195.

DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403