

## SUMMABILITY INTEGRALS

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**ABSTRACT.** An integral on the set of natural numbers  $N$  is defined. If  $E$  is a subset of  $N$  such that the characteristic function  $\chi_E$  of  $E$  is evaluated to some number  $\sigma$  by a regular nonnegative summation matrix  $A$ , then  $\mu_A(E)$  is defined to be  $\sigma$ ;  $\mu_A$  defines a finitely additive measure on  $N$ . If  $s$  is a sequence which can be written as a linear combination of characteristic functions  $\sum_{i=1}^n a_i \chi_{E_i}$ , where each sequence  $\chi_{E_i}$  is evaluated by  $A$ , then  $\int_N s d\mu_A$  is defined to be  $\sum a_i \mu_A(E_i)$ . Finally the definition of the integral is naturally extended to  $L(A)$ , the class of sequences which can be approximated by linear combinations of characteristic functions [2, pp. 85-88]. It is shown that if  $A$  and  $B$  are two nonnegative regular matrices such that the convergence field of  $A$  includes that of  $B$ , then  $L(A)$  includes  $L(B)$  provided  $B$  is normal. Finally for a nonnegative regular matrix  $A = (a_{nk})$ , the spaces  $L^p(A)$  spanned by sequences such that  $\{\sum_{k=0}^{\infty} a_{nk} |s_k|^p\}$  is bounded and  $\lim \sum_{k=0}^{\infty} a_{nk} s_k^p$  exists are studied. It is shown that if  $p$  is greater than one, then the sequences in  $L^p(A)$  give rise to a set of bounded linear functionals on  $L^p(A)$  which are weak star dense in the dual of  $L^p(A)$ .

Let  $A = (a_{nk})$  be a nonnegative regular summation matrix and let  $\Sigma = \Sigma_A$  denote the class of subsets  $E$  of the natural numbers  $N$  such that

$$\lim_{n \rightarrow \infty} \sum_{k \in E} a_{nk} = \mu_A(E)$$

exists, that is,  $\Sigma$  denotes the class of sets  $E$  whose characteristic functions  $\chi_E$  are  $A$ -summable; thus  $\mu_A$  is a finitely additive measure on  $N$  (the characteristic function of  $E$  is the sequence which is one when  $n$  is in  $E$  and zero when  $n$  is not in  $E$ ). Let  $S(\Sigma_A)$  denote the set of finite linear combinations  $\sum a_i \chi_{E_i}$ , where the  $a_i$  are scalars and the sets  $E_i$  are in  $\Sigma$ . Each sequence  $s$  in  $S(\Sigma)$  lies in the convergence field,  $W(A)$ , of  $A$ ; if we write

$$\int_N s d\mu_A = \sum_{i=1}^n a_i \mu_{E_i}(A)$$

for each sequence  $s = \sum a_i \chi_{E_i}$  in  $S(\Sigma)$ ,  $E_i \in \Sigma$ , then

$$\int_N s d\mu_A = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} s_k.$$

Edwards and Wayment [1, p. 88] define the summability integral as follows:

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the norm of a sequence  $s$  in  $S(\Sigma)$  is defined by

$$(1) \quad \|s\| = \|s\|_A = \limsup_n \sum_{k=0}^{\infty} a_{nk} |s_k|.$$

$L(A)$  denotes the closure of  $S(\Sigma_A)$  in the norm (1). If  $s \in L(A)$ , then for each  $\varepsilon > 0$  there exists a sequence  $s_\varepsilon$  in  $S(\Sigma)$  such that  $\|s - s_\varepsilon\| \leq \varepsilon$ ; we define

$$(2) \quad \int s \, d\mu_A = \lim_{\varepsilon \rightarrow 0} \int s_\varepsilon \, d\mu_A.$$

Clearly  $\int s \, d\mu_A$  defined above is independent of the choice of  $s_\varepsilon$ . We norm  $L(A)$  by

$$(1') \quad \|s\| = \|s\|_A = \limsup_n \sum_{k=0}^{\infty} a_{nk} |s_k|.$$

Two sequences  $s$  and  $t$  are identified in  $L(A)$  whenever  $\|s - t\| = 0$ . Formula (1') extends the definition of the norm from  $S(\Sigma)$  to  $L(A)$ .

For each  $E \subseteq N$  the symbol  $\bar{\mu}_A(E)$  will denote  $\limsup \sum_{k \in E} |a_{nk}|$ ;  $\mu_A$  is an outer measure, and  $\bar{\mu}_A(E) = \|\chi_E\|$  if  $\chi_E \in L(A)$ .

Edwards and Waymott [1, p. 92] ask whether  $L(A) \supseteq L(B)$  implies  $W(A) \supseteq W(B)$ , where  $W(A)$  denotes the convergence field of the summation matrix  $A$ . We will exhibit two nonnegative regular matrices  $A$  and  $B$  such that  $L(A) \supseteq L(B)$  while  $W(A) \not\supseteq W(B)$ . Let  $A$  be given by the equations

$$\begin{aligned} a_{n,n} &= 1, & a_{n,k} &= 0 & \text{when } k \neq n, n \text{ even,} \\ a_{n,n-1} &= 1, & a_{n,k} &= 0 & \text{when } k \neq n-1, n \text{ odd,} \end{aligned}$$

and let  $B$  be the Nörlund matrix given by the equations

$$\begin{aligned} b_{00} &= 1, & b_{0k} &= 0, & k \neq 0, \\ b_{n,n} &= b_{n,n-1} = 1/2, \\ b_{n,k} &= 0, & k &\neq n, k \neq n-1, n \geq 1. \end{aligned}$$

The only sequences  $s$  in  $L(B)$  are those for which both subsequences  $\{s_{2n}\}$  and  $\{s_{2n+1}\}$  converge, while  $L(A)$  contains all sequences  $t$  such that  $\{t_{2n}\}$  converges; hence  $L(A) \supseteq L(B)$ . However,  $A$  evaluates only those sequences  $s$  such that the subsequence  $\{s_{2n}\}$  converges, while  $B$  evaluates all sequences  $s$  such that  $\{s_n + s_{n+1}\}$  converges. Hence  $W(A) \not\supseteq W(B)$ .

In the reverse direction we have

**THEOREM.** *If  $A$  and  $B$  are nonnegative regular matrices,  $B$  is normal and  $W(A) \supseteq W(B)$ , then  $L(A) \supseteq L(B)$ .*

A normal matrix is a triangular matrix with all diagonal elements nonzero.

**PROOF.** Suppose that  $s$  is a sequence in  $L(B)$ . Then for each  $\varepsilon > 0$  there exist sets  $E_i \in \Sigma_B$  ( $i = 1, 2, \dots, r$ ) and scalars  $a_i$  such that  $\|s - \sum_{i=1}^r a_i \chi_{E_i}\|_B \leq \varepsilon$ . Since each sequence  $\chi_{E_i}$  is evaluated by  $B$ , it is evaluated by  $A$ , that is,  $E_i \in \Sigma_A$  for each  $i$ . Since  $B$  is normal,  $B^{-1}$  exists. Our hypothesis  $W(A) \supseteq W(B)$  implies that the matrix  $AB^{-1}$  is regular. Thus  $AB^{-1}$  represents a bounded operator on the Banach space  $m_0$  of bounded sequences, normed by

$$\|s\| = \limsup |s_n|, \quad s \in m_0.$$

(Sequences differing by a null sequence are identified in  $m_0$ .) Let  $u = |B^{-1}(s - \sum_{i=1}^r A_i \chi_{E_i})|$ , that is,

$$u_n = \left| \left\{ B^{-1} \left( s - \sum_{i=1}^r a_i \chi_{E_i} \right) \right\}_n \right|, \quad n = 0, 1, 2, \dots$$

We have  $\|Bu\| = \|u\|_B \leq \varepsilon$ , and consequently  $\|Au\| \leq \|AB^{-1}\|\varepsilon$ , where  $\|AB^{-1}\|$  denotes the norm of the operator on  $m_0$  represented by the matrix  $AB^{-1}$ . (Again, since  $AB^{-1}$  is regular,  $\|AB^{-1}\| < \infty$ .) But

$$\|Au\| = \|s - \sum a_i \chi_{E_i}\|_A$$

and therefore this quantity is bounded by  $\|AB^{-1}\|\varepsilon$ . Since each set  $E_i$  lies in  $\Sigma_A$ ,  $s \in L(A)$ . This completes the proof.

We define, for a nonnegative regular summation matrix  $A = (a_{nk})$ ,  $p > 1$ , the space  $L^p(A)$  consisting of linear combinations of sequences  $s = \{s_n\}$  such that

$$(8_p) \quad \|s\|_p = \limsup \left( \sum_{k=0}^{\infty} a_{nk} |s_k|^p \right)^{1/p} < \infty,$$

and

$$(9_p) \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} s_k^p$$

exists. We note that  $(8_p)$  defines a norm on  $L^p(A)$ . Edwards and Wayment [1, p. 92] raise the question whether the dual of  $L^p(A)$  is  $L^{p'}(A)$ ,  $p' = p/(p - 1)$ . In general each sequence  $t \in L^{p'}(A)$  gives rise to infinitely many bounded functionals on  $L^p(A)$ ,  $p > 1$ . Suppose that  $s \in L^p(A)$  and  $t \in L^{p'}(A)$ . Then  $s$  satisfies  $(8_p)$  while  $t$  satisfies  $(8_{p'})$ . The sequence  $\{\sum_{k=0}^{\infty} a_{nk} s_k t_k\}$  is bounded; as a bounded function on the discrete space  $N$  it has a continuous extension  $\{\sum_{k=0}^{\infty} a_{nk} s_k t_k\}^\beta$  to  $\beta N$ , the Stone-Ćech compactification of  $N$ . For a description of the Stone-Ćech compactification, the reader is referred to [2, pp. 82-89]. (Throughout the paper the superscript  $\beta$  will indicate continuous extension to  $\beta N$ ; for each point  $\gamma \in \beta N - N$  we will indicate that the function  $\{\sum_{k=j}^{\infty} a_{nk} s_k t_k\}^\beta$  has been evaluated at  $\gamma$  by writing  $\{\sum a_{nk} s_k t_k\}_\gamma^\beta$ .) If  $s$  and  $t$  are sequences such that  $A$  fails to evaluate  $st$ , then  $t$  gives rise to bounded functionals

$$L(t, \gamma)s = \left\{ \sum_{k=0}^{\infty} a_{nk} s_k t_k \right\}_\gamma^\beta, \quad \gamma \in \beta N - N;$$

the functionals  $L(t, \gamma)(s)$  take on different values as  $\gamma$  ranges over the points of  $\beta N - N$ .

**THEOREM.** *If  $A$  is a nonnegative regular matrix and  $p > 1$ , then the functionals*

$$(10) \quad L(t, \gamma)(s) = \left( \sum_{k=0}^{\infty} a_{nk} s_k t_k \right)_\gamma^\beta, \quad s \in L^p(A),$$

where  $t$  ranges over  $L^{p'}(A)$  and  $\gamma$  ranges over  $\beta N - N$ , are weak star dense in the dual of  $L^p(A)$ .

Recall that the superscript  $\beta$  in (10) indicates that the function  $\{\sum a_{nk} s_k t_k\}$  has been continuously extended to  $\beta N - N$ ; the subscript  $\gamma$  indicates that the extended function has been evaluated at the point  $\gamma$ .

PROOF. We first treat the case  $p = 2$ . Let  $s$  be a sequence in  $L^2(A)$ ; then  $\bar{s} = \{\bar{s}_n\}$  is in  $L^2(A)$  and  $\limsup \sum_{k=0}^{\infty} a_{nk} s_k \bar{s}_k = \|s\|_2^2$ . There is a point  $\gamma \in \beta N - N$  such that  $L(\bar{s}, \gamma)(s) = \|s\|_2^2$ . Hence if  $L(t, \gamma)s = 0$  for all sequences  $t \in L^2(A)$  and all points  $\gamma \in \beta N - N$ , then  $\|s\|_2 = 0$ ; that is, for  $p = 2$ , functionals of the form (10) are dense in the dual of  $L^p(A)$ .

For the other cases we need the following lemma whose proof is obvious.

LEMMA. Let  $0 < \alpha < 2\pi$ ,  $\alpha \neq \pi$  and let  $\theta$  be a real number. If  $\alpha$  is not a rational multiple of  $2\pi$ , then for each  $\epsilon > 0$  there exists an integer  $m$  such that  $|\theta - m\alpha| < \epsilon \pmod{2\pi}$ . If  $\alpha$  is a rational multiple of  $2\pi$ , then there exists an integer  $m$  such that  $|\theta - m\alpha| < \alpha/2 \pmod{2\pi}$ .

We return to the proof of the theorem. Let  $s$  be a sequence in  $L^p(A)$ ; it is sufficient to show that if  $s$  is a sequence in  $L^p(A)$  and  $\sum_{k=0}^{\infty} a_{nk} s_k t_k$  tends to zero for each sequence  $t \in L^{p'}(A)$ , then  $\|s\|_p = 0$ . Suppose that  $\|s\|_p > 0$ . If  $s \in L^p(A)$ , then  $s$  may be written  $s = \sum_{i=1}^r s^{(i)}$  where each sequence  $s^{(i)}$  satisfies (8<sub>p</sub>) and (9<sub>p</sub>). The sequences  $(s^{(i)})^{p-1}$  ( $i = 1, 2, \dots, r$ ) satisfy (8<sub>p</sub>) and (9<sub>p</sub>) and hence the sequence  $\bar{s} = \sum_{i=1}^r (s^{(i)})^{p-1}$  is in  $L^{p'}(A)$  as is each sequence

$$t(m) = \{t_k(m)\} = \{\bar{s}_k^{p-1} \exp 2m\pi i/p'\}, \quad m = 0, 1, \dots$$

The case  $2\pi/p' = \pi$  can occur only if  $p = 2$ , and this case has been dealt with. Hence we may apply the lemma with  $\alpha = 2\pi/p' \neq \pi$  and  $\theta$  as the angle between the vector  $\bar{s}_k^{p-1}$  and the vector  $u_k = |s_k|^{p-2} \bar{s}_k$  to conclude that if  $p$  is irrational, then for each  $\epsilon > 0$  there exists an integer  $m_k$  such that the angle between the vector  $t_k(m_k)$  and the vector  $u_k$  is numerically less than  $\epsilon$  whenever  $s_k \neq 0$ , while if  $p$  is rational,  $p \neq 2$ , then the angle between the vector  $t_k(m_k)$  and the vector  $u_k$  is numerically no greater than  $|\pi - 2\pi/p'| = \pi/p'$  wherever  $s_k \neq 0$ . For such  $m_k$ ,

$$\operatorname{Re}|s_k t_k(m_k)| \geq |s_k|^p \cos \epsilon$$

if  $p$  is irrational, while

$$\operatorname{Re}|s_k t_k(m_k)| \geq |s_k|^p \cos \pi/p'$$

if  $p$  is rational. Hence

$$\limsup \sum_{k=0}^{\infty} a_{nk} s_k t_k(m_k) \geq \|s_k\|_p^p \cos \epsilon$$

if  $p$  is irrational, while

$$\limsup \sum_{k=0}^{\infty} a_{nk} s_k t_k(m_k) \geq \|s\|_p^p \cos \pi/p'$$

if  $p$  is rational,  $p \neq 2$ . In either case if  $\|s\|_p > 0$ , then  $\limsup \sum_{k=0}^{\infty} a_{nk} s_k t_k(m_k) > 0$ . This concludes the proof.

The proof actually shows

THEOREM. If  $s \in L^p(A)$ ,  $\|s\|_p > 0$  where  $A$  is a nonnegative regular matrix and  $p > 1$ , then for  $p = 2$  there exists a sequence  $u \in L^2(A)$  such that  $\|u\|_2$

$= 1$  and  $\limsup \sum a_{nk} s_k u_k = \|s\|_2$ ; for  $p$  irrational there exists, for each positive  $\delta$ , a sequence  $u \in L^{p'}(A)$  such that  $\|u\|_{p'} = 1$  and

$$\limsup \sum a_{nk} s_k u_k \geq (1 - \delta) \|s\|_p;$$

for  $p$  rational,  $p \neq 2$ , there exists a sequence  $u \in L^{p'}(A)$  such that  $\|u\|_{p'} = 1$  and  $\limsup \sum a_{nk} s_k u_k \geq \|s\|_p \cos \pi/p'$ .

## REFERENCES

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