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SUMMABILITY INTEGRALS

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ABSTRACT. An integral on the set of natural numbers N is defined. If E is a subset of N such that the characteristic function χ_E of E is evaluated to some number σ by a regular nonnegative summation matrix A, then $\mu_A(E)$ is defined to be σ; μ_A defines a finitely additive measure on N. If s is a sequence which can be written as a linear combination of characteristic functions $\sum_{i=1}^n a_i \chi_{E_i}$, where each sequence χ_{E_i} is evaluated by A, then $\int_N s\, d\mu_A$ is defined to be $\sum a_i \mu_A(E_i)$. Finally the definition of the integral is naturally extended to $L(A)$, the class of sequences which can be approximated by linear combinations of characteristic functions [2, pp. 85–88]. It is shown that if A and B are two nonnegative regular matrices such that the convergence field of A includes that of B, then $L(A)$ includes $L(B)$ provided B is normal. Finally for a nonnegative regular matrix $A = (a_{nk})$, the spaces $L^p(A)$ spanned by sequences such that $\{\sum_{k=0}^\infty a_{nk}|s_k|^p\}$ is bounded and $\lim \sum_{k=0}^\infty a_{nk} s_k^p$ exists are studied. It is shown that if p is greater than one, then the sequences in $L^{p'}(A)$ give rise to a set of bounded linear functionals on $L^p(A)$ which are weak star dense in the dual of $L^p(A)$.

Let $A = (a_{nk})$ be a nonnegative regular summation matrix and let $\Sigma = \Sigma_A$ denote the class of subsets E of the natural numbers N such that

$$\lim_{n\to\infty} \sum_{k \in E} a_{nk} = \mu_A(E)$$

exists, that is, Σ denotes the class of sets E whose characteristic functions χ_E are A-summable; thus μ_A is a finitely additive measure on N (the characteristic function of E is the sequence which is one when n is in E and zero when n is not in E). Let $S(\Sigma_A)$ denote the set of finite linear combinations $\sum a_i \chi_{E_i}$, where the a_i are scalars and the sets E_i are in Σ. Each sequence s in $S(\Sigma)$ lies in the convergence field, $W(A)$, of A; if we write

$$\int_N s\, d\mu_A = \sum_{i=1}^n a_i \mu_{E_i}(A)$$

for each sequence $s = \sum a_i \chi_{E_i}$ in $S(\Sigma)$, $E_i \in \Sigma$, then

$$\int_N s\, d\mu_A = \lim_{n\to\infty} \sum_{k=0}^\infty a_{nk} s_k.$$

Edwards and Wayment [1, p. 88] define the summability integral as follows:

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the norm of a sequence s in $S(\Sigma)$ is defined by

$$\|s\| = \|s\|_A = \limsup_n \sum_{k=0}^{\infty} a_{nk}|s_k|. \tag{1}$$

$L(A)$ denotes the closure of $S(\Sigma_A)$ in the norm (1). If $s \in L(A)$, then for each $\varepsilon > 0$ there exists a sequence s_ε in $S(\Sigma)$ such that $\|s - s_\varepsilon\| \leqslant \varepsilon$; we define

$$\int s \, d\mu_A = \lim_{\varepsilon \to 0} \int s_\varepsilon \, d\mu_A. \tag{2}$$

Clearly $\int s \, d\mu_A$ defined above is independent of the choice of s_ε. We norm $L(A)$ by

$$\|s\| = \|s\|_A = \limsup_n \sum_{k=0}^{\infty} a_{nk}|s_k|. \tag{1'}$$

Two sequences s and t are identified in $L(A)$ whenever $\|s - t\| = 0$. Formula (1′) extends the definition of the norm from $S(\Sigma)$ to $L(A)$.

For each $E \subseteq N$ the symbol $\bar{\mu}_A(E)$ will denote $\limsup \Sigma_{k \in E} |a_{nk}|$; μ_A is an outer measure, and $\bar{\mu}_A(E) = \|\chi_E\|$ if $\chi_E \in L(A)$.

Edwards and Wayment [1, p. 92] ask whether $L(A) \supseteq L(B)$ implies $W(A) \supseteq W(B)$, where $W(A)$ denotes the convergence field of the summation matrix A. We will exhibit two nonnegative regular matrices A and B such that $L(A) \supseteq L(B)$ while $W(A) \not\supseteq W(B)$. Let A be given by the equations

$$a_{n,n} = 1, \quad a_{n,k} = 0 \quad \text{when } k \neq n, \ n \text{ even},$$
$$a_{n,n-1} = 1, \quad a_{n,k} = 0 \quad \text{when } k \neq n-1, \ n \text{ odd},$$

and let B be the Nörlund matrix given by the equations

$$b_{00} = 1, \quad b_{0k} = 0, \qquad k \neq 0,$$
$$b_{n,n} = b_{n,n-1} = 1/2,$$
$$b_{n,k} = 0, \qquad k \neq n, \ k \neq n-1, \ n \geqslant 1.$$

The only sequences s in $L(B)$ are those for which both subsequences $\{s_{2n}\}$ and $\{s_{2n+1}\}$ converge, while $L(A)$ contains all sequences t such that $\{t_{2n}\}$ converges; hence $L(A) \supseteq L(B)$. However, A evaluates only those sequences s such that the subsequence $\{s_{2n}\}$ converges, while B evaluates all sequences s such that $\{s_n + s_{n+1}\}$ converges. Hence $W(A) \not\supseteq W(B)$.

In the reverse direction we have

THEOREM. *If A and B are nonnegative regular matrices, B is normal and $W(A) \supseteq W(B)$, then $L(A) \supseteq L(B)$.*

A normal matrix is a triangular matrix with all diagonal elements nonzero.

PROOF. Suppose that s is a sequence in $L(B)$. Then for each $\varepsilon > 0$ there exist sets $E_i \in \Sigma_B$ $(i = 1, 2, \ldots, r)$ and scalars a_i such that $\|s - \Sigma_{i=1}^r a_i \chi_{E_i}\|_B \leqslant \varepsilon$. Since each sequence χ_{E_i} is evaluated by B, it is evaluated by A, that is, $E_i \in \Sigma_A$ for each i. Since B is normal, B^{-1} exists. Our hypothesis $W(A) \supseteq W(B)$ implies that the matrix AB^{-1} is regular. Thus AB^{-1} represents a bounded operator on the Banach space m_0 of bounded sequences, normed by

$$|||s||| = \limsup_n |s_n|, \qquad s \in m_0.$$

(Sequences differing by a null sequence are identified in m_0.) Let $u = |B^{-1}(s - \Sigma_{i=1}^{r} A_i \chi_{E_i})|$, that is,

$$u_n = \left| \left\{ B^{-1} \left(s - \sum_{i=1}^{r} a_i \chi_{E_i} \right) \right\}_n \right|, \qquad n = 0, 1, 2, \ldots.$$

We have $|||Bu||| = \|u\|_B \leqslant \varepsilon$, and consequently $|||Au||| \leqslant |||AB^{-1}|||\varepsilon$, where $|||AB^{-1}|||$ denotes the norm of the operator on m_0 represented by the matrix AB^{-1}. (Again, since AB^{-1} is regular, $|||AB^{-1}||| < \infty$.) But

$$|||Au||| = \|s - \Sigma a_i \chi_{E_i}\|_A$$

and therefore this quantity is bounded by $|||AB^{-1}|||\varepsilon$. Since each set E_i lies in Σ_A, $s \in L(A)$. This completes the proof.

We define, for a nonnegative regular summation matrix $A = (a_{nk})$, $p > 1$, the space $L^p(A)$ consisting of linear combinations of sequences $s = \{s_n\}$ such that

$$(8_p) \qquad \|s\|_p = \limsup \left(\sum_{k=0}^{\infty} a_{nk} |s_k|^p \right)^{1/p} < \infty,$$

and

$$(9_p) \qquad \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} s_k^p$$

exists. We note that (8_p) defines a norm on $L^p(A)$. Edwards and Wayment [1, p. 92] raise the question whether the dual of $L^p(A)$ is $L^{p'}(A)$, $p' = p/(p-1)$. In general each sequence $t \in L^{p'}(A)$ gives rise to infinitely many bounded functionals on $L^p(A)$, $p > 1$. Suppose that $s \in L^p(A)$ and $t \in L^{p'}(A)$. Then s satisfies (8_p) while t satisfies $(8_{p'})$. The sequence $\{\Sigma_{k=0}^{\infty} a_{nk} s_k t_k\}$ is bounded; as a bounded function on the discrete space N it has a continuous extension $\{\Sigma_{k=0}^{\infty} a_{nk} s_k t_k\}^{\beta}$ to βN, the Stone-Čech compactification of N. For a description of the Stone-Čech compactification, the reader is referred to [2, pp. 82-89]. (Throughout the paper the superscript β will indicate continuous extension to βN; for each point $\gamma \in \beta N - N$ we will indicate that the function $\{\Sigma_{k=j}^{\infty} a_{nk} s_k t_k\}^{\beta}$ has been evaluated at γ by writing $\{\Sigma a_{nk} s_k t_k\}_{\gamma}^{\beta}$.) If s and t are sequences such that A fails to evaluate st, then t gives rise to bounded functionals

$$L(t, \gamma)s = \left\{ \sum_{k=0}^{\infty} a_{nk} s_k t_k \right\}_{\gamma}^{\beta}, \qquad \gamma \in \beta N - N;$$

the functionals $L(t, \gamma)(s)$ take on different values as γ ranges over the points of $\beta N - N$.

THEOREM. *If A is a nonnegative regular matrix and $p > 1$, then the functionals*

$$(10) \qquad L(t, \gamma)(s) = \left(\sum_{k=0}^{\infty} a_{nk} s_k t_k \right)_{\gamma}^{\beta}, \qquad s \in L^p(A),$$

where t ranges over $L^{p'}(A)$ and γ ranges over $\beta N - N$, are weak star dense in the dual of $L^p(A)$.

Recall that the superscript β in (10) indicates that the function $\{\Sigma a_{nk}s_k t_k\}$ has been continuously extended to $\beta N - N$; the subscript γ indicates that the extended function has been evaluated at the point γ.

PROOF. We first treat the case $p = 2$. Let s be a sequence in $L^2(A)$; then $\bar{s} = \{\bar{s}_n\}$ is in $L^2(A)$ and $\lim\sup\Sigma_{k=0}^{\infty}a_{nk}s_k\bar{s}_k = \|s\|_2^2$. There is a point $\gamma \in \beta N - N$ such that $L(\bar{s}, \gamma)(s) = \|s\|_2^2$. Hence if $L(t, \gamma)s = 0$ for all sequences $t \in L^2(A)$ and all points $\gamma \in \beta N - N$, then $\|s\|_2 = 0$; that is, for $p = 2$, functionals of the form (10) are dense in the dual of $L^p(A)$.

For the other cases we need the following lemma whose proof is obvious.

LEMMA. *Let $0 < \alpha < 2\pi$, $\alpha \neq \pi$ and let θ be a real number. If α is not a rational multiple of 2π, then for each $\varepsilon > 0$ there exists an integer m such that $|\theta - m\alpha| < \varepsilon$ (mod 2π). If α is a rational multiple of 2π, then there exists an integer m such that $|\theta - m\alpha| < \alpha/2$ (mod 2π).*

We return to the proof of the theorem. Let s be a sequence in $L^p(A)$; it is sufficient to show that if s is a sequence in $L^p(A)$ and $\Sigma_{k=0}^{\infty}a_{nk}s_k t_k$ tends to zero for each sequence $t \in L^{p'}(A)$, then $\|s\|_p = 0$. Suppose that $\|s\|_p > 0$. If $s \in L^p(A)$, then s may be written $s = \Sigma_{i=1}^{r}s^{(i)}$ where each sequence $s^{(i)}$ satisfies (8_p) and (9_p). The sequences $(\bar{s}^{(i)})^{p-1}$ $(i = 1, 2, \ldots, r)$ satisfy ($8_{p'}$) and ($9_{p'}$) and hence the sequence $\bar{s} = \Sigma_{i=1}^{r}(\bar{s}^{(i)})^{p-1}$ is in $L^{p'}(A)$ as is each sequence

$$t(m) = \{t_k(m)\} = \{\bar{s}_k^{p-1}\exp 2m\pi i/p'\}, \qquad m = 0, 1, \ldots.$$

The case $2\pi/p' = \pi$ can occur only if $p = 2$, and this case has been dealt with. Hence we may apply the lemma with $\alpha = 2\pi/p' \neq \pi$ and θ as the angle between the vector $\bar{s}_k^{p-1}$ and the vector $u_k = |s_k|^{p-2}\bar{s}_k$ to conclude that if p is irrational, then for each $\varepsilon > 0$ there exists an integer m_k such that the angle between the vector $t_k(m_k)$ and the vector u_k is numerically less than ε whenever $s_k \neq 0$, while if p is rational, $p \neq 2$, then the angle between the vector $t_k(m_k)$ and the vector u_k is numerically no greater than $|\pi - 2\pi/p'| = \pi/p'$ wherever $s_k \neq 0$. For such m_k,

$$\operatorname{Re}|s_k t_k(m_k)| \geqslant |s_k|^p \cos\varepsilon$$

if p is irrational, while

$$\operatorname{Re}|s_k t_k(m_k)| \geqslant |s_k|^p \cos\pi/p'$$

if p is rational. Hence

$$\lim\sup\sum_{k=0}^{\infty} a_{nk}s_k t_k(m_k) \geqslant \|s_k\|_p^p \cos\varepsilon$$

if p is irrational, while

$$\lim\sup\sum_{k=0}^{\infty} a_{nk}s_k t_k(m_k) \geqslant \|s\|_p^p \cos\pi/p'$$

if p is rational, $p \neq 2$. In either case if $\|s\|_p > 0$, then $\lim\sup\Sigma_{k=0}^{\infty}a_{nk}s_k t_k(m_k) > 0$. This concludes the proof.

The proof actually shows

THEOREM. *If $s \in L^p(A)$, $\|s\|_p > 0$ where A is a nonnegative regular matrix and $p > 1$, then for $p = 2$ there exists a sequence $u \in L^2(A)$ such that $\|u\|_2$*

$= 1$ *and* $\limsup \sum a_{nk} s_k u_k = \|s\|_2$; *for* p *irrational there exists, for each positive* δ, *a sequence* $u \in L^{p'}(A)$ *such that* $\|u\|_{p'} = 1$ *and*

$$\limsup \sum a_{nk} s_k u_k \geqslant (1 - \delta)\|s\|_p;$$

for p *rational*, $p \neq 2$, *there exists a sequence* $u \in L^{p'}(A)$ *such that* $\|u\|_{p'} = 1$ *and* $\limsup \sum a_{nk} s_k u_k \geqslant \|s\|_p \cos \pi / p'$.

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