PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

DEPENDENCE ON DIMENSION OF A CONSTANT RELATED TO THE GRÖTZSCH RING

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ABSTRACT. For the constant $\lambda_n = \lim_{a\to 0}(\text{mod } R_{G,n}(a) + \log a)$ associated with the Grötzsch extremal ring $R_{G,n}$ in euclidean n-space, we obtain the limit $\lim_{n\to\infty} \lambda_n^{1/n} = e$.

1. **Definitions and notation.** By a *ring* R is meant a domain in euclidean n-space R^n whose complement consists of two components C_0 and C_1, where C_0 is bounded. We let $B_0 = \partial C_0$ and $B_1 = \partial C_1$ be the boundary components of R. The *conformal capacity* (cf. [8]) of R is

$$\text{cap } R = \inf_{\varphi} \int_R |\nabla\varphi|^n \, d\omega,$$

where ∇ denotes the gradient, and where the infimum is taken over all real-valued C^1 functions φ in R with boundary values 0 on B_0 and 1 on B_1. Then the *modulus* of the ring R is defined by

$$\text{mod } R = (\sigma_{n-1}/\text{cap } R)^{1/(n-1)},$$

where for each positive integer p we let σ_p denote the p-dimensional measure of the unit sphere $S^p = \{(x_1, \ldots, x_{p+1}): \sum_{j=1}^{p+1} x_j^2 = 1\}$. Then

$$\sigma_p = 2\pi^{(p+1)/2}\Gamma((p+1)/2)^{-1}$$

(cf. [7], [9]), where Γ denotes the classical Gamma function; moreover, the relation

$$\int_0^{\pi/2} \cos^p u \, du = \frac{\sigma_{p+1}}{2\sigma_p} \tag{1}$$

holds for each positive integer p.

2. **Background of problem and statement of result.** Let $R_{G,n}(a)$ denote the n-dimensional Grötzsch ring, that is, the ring whose complementary components are

$$C_0 = \{(x_1, \ldots, x_n): 0 \leqslant x_1 \leqslant a, x_j = 0, 2 \leqslant j \leqslant n\}$$

and

Received by the editors June 2, 1975.
AMS (MOS) subject classifications (1970). Primary 30A60.

$$C_1 = \left\{ (x_1, \ldots, x_n) \colon \sum_{j=1}^{n} x_j^2 \geqslant 1 \right\}.$$

In [5] Gehring proved that mod $R_{G,3}(a) + \log a$ is monotone decreasing in the interval $0 < a < 1$ and he obtained bounds $4 \leqslant \lambda_3 \leqslant 12.4 \cdots$, where $\log \lambda_3 = \lim_{a \to 0}(\text{mod } R_{G,3}(a) + \log a)$. Using analogous methods in higher dimensions, Caraman [3] and Ikoma [7] have shown that the limit

$$\log \lambda_n = \lim_{a \to 0} (\text{mod } R_{G,n}(a) + \log a)$$

exists for each $n \geqslant 3$. Bounds for λ_n have been obtained by Gehring [5], [6], Caraman [3], Ikoma [7], and Anderson [2] (cf. [1]).

In the present paper we determine the order of growth of λ_n for large n by establishing the following result.

THEOREM. $\lim_{n \to \infty} \lambda_n^{1/n} = e$, *where* e *is the base of Naperian logarithms.*

3. **Proof of Theorem.** First, by the work of Caraman [3] and Ikoma [7] we know that

$$\log \frac{\lambda_n}{4} \leqslant I_n = \int_1^{\infty} \left[\left(\frac{r^2+1}{r^2-1} \right)^{(n-2)/(n-1)} - 1 \right] \frac{dr}{r}. \tag{2}$$

Under the change of variable $t = (r^2 + 1)/(r^2 - 1)$ the integral I_n in (2) reduces to

$$I_n = \int_1^{\infty} \frac{t^{(n-2)/(n-1)} - 1}{t^2 - 1} \, dt. \tag{3}$$

Making use of the fact that $(t^2 - 1)^{-1} = t^{-2} + t^{-2}(t^2 - 1)^{-1}$, one may rewrite I_n as the sum of two integrals. The first of these is easy to evaluate exactly as $n - 2$, and by means of the Monotone Convergence Theorem it is easy to see that the second increases to the limit $1 - \log 2$. This procedure, together with (2), yields the result

$$\log \frac{\lambda_n}{4} - n \leqslant \lim_{n \to \infty} (I_n - n) = -1 - \log 2, \tag{4}$$

from which the upper bound $\lambda_n^{1/n} \leqslant e$ is easily derived (cf. [4, Lecture 9, Theorem 1]).

On the other hand, it was determined in [2] that

$$\log \frac{\lambda_n}{4} \geqslant \int_0^{\infty} \left[\left(\frac{2\sigma_{n-2}}{\sigma_{n-1}} \int_0^{\pi/2} (\sec^2 u + \operatorname{csch}^2 v)^{(2-n)/2} du \right)^{1/(1-n)} - 1 \right] dv. \tag{5}$$

By making the change of variable $t = \coth v$ in (5) and combining the resulting inequality with (1), (2), and (3), we achieve

$$0 \geqslant \log \frac{\lambda_n}{4} - I_n \geqslant \int_1^{\infty} \Phi_n(t) \, dt, \tag{6}$$

where

$$(7)\quad \Phi_n(t) = \frac{t^{(n-2)/(n-1)}}{t^2-1}\left[\left(\frac{2\sigma_{n-2}}{\sigma_{n-1}}\int_0^{\pi/2}\left(1+\frac{\tan^2 u}{t^2}\right)^{(2-n)/2} du\right)^{1/(1-n)} - 1\right].$$

We consider first the integral of $\Phi_n(t)$ over the finite interval $(1, M)$. First, using (1), it is convenient to rewrite (7) in the form

$$(8)\quad \Phi_n(t) = \frac{t^{(n-2)/(n-1)}}{t^2-1}\left[\left(\frac{\|\cos u\|_{n-2}}{\|(1+t^{-2}\tan^2 u)^{-1/2}\|_{n-2}}\right)^{(n-2)/(n-1)} - 1\right],$$

where $\|f\|_p$ denotes the $L_p[0,\pi/2]$-norm of f. Since the functions $\cos u$ and $(1+t^{-2}\tan^2 u)^{-1/2}$ are continuous on $[0,\pi/2]$, a well-known result in the theory of L_p spaces gives

$$(9)\quad \lim_{n\to\infty} \frac{\|\cos u\|_{n-2}}{\|(1+t^{-2}\tan^2 u)^{-1/2}\|_{n-2}} = \frac{\max_{0\leqslant u\leqslant \pi/2}|\cos u|}{\max_{0\leqslant u\leqslant \pi/2}|1+t^{-2}\tan^2 u|^{-1/2}} = 1.$$

From (8) and (9) it follows that for each fixed $t \in (1, M)$,

$$(10)\quad \lim_{n\to\infty} \Phi_n(t) = 0.$$

Next, in view of (8) it is easy to see that

$$0 > \Phi_n(t) \geqslant t^{(n-2)/(n-1)}(t^{(2-n)/(n-1)} - 1)/(t^2-1) \geqslant -1/(1+t).$$

This and (10) allow us to invoke the Dominated Convergence Theorem to conclude that

$$(11)\quad \lim_{n\to\infty}\int_1^M \Phi_n(t)\,dt = 0$$

for each fixed $M \in (1,\infty)$.

Next, using the fact that $t^2/(t^2-1)$ is decreasing for $t > 1$ and that $\Phi_n(t) \leqslant 0$, we may write

$$\int_M^\infty \Phi_n(t)\,dt \geqslant \frac{M^2}{M^2-1}\left[\left(\frac{\|\cos u\|_{n-2}}{\|1\|_{n-2}}\right)^{(n-2)/(n-1)} - 1\right]\int_M^\infty t^{(n-2)/(n-1)-2}\,dt.$$

Thus

$$\frac{1}{n}\int_M^\infty \Phi_n(t)dt \geqslant \frac{M^2}{M^2-1}\left(1-\frac{1}{n}\right)M^{1/(1-n)}\left[\left(\frac{2}{\pi}\right)^{1/(n-1)}\|\cos u\|_{n-2}^{(n-2)/(n-1)} - 1\right],$$

and the same argument as that leading to (10) shows that

$$\liminf_{n\to\infty} \frac{1}{n}\int_M^\infty \Phi_n(t)\,dt \geqslant 0.$$

But because of (11), (6), and (4), this implies the statement

$$(12)\quad \liminf_{n\to\infty} \frac{1}{n}\log\lambda_n \geqslant 1.$$

Finally, the Theorem follows from (2) and (12).

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