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ON PROXIMINAL SETS OF NORMAL OPERATORS

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ABSTRACT. It is shown that neither the set of normal operators nor the set of orthogonal projections is a proximinal subset of the space of bounded operators on an infinite-dimensional Hilbert space.

Let H denote a separable complex Hilbert space with inner product $(\ ,\ )$ and norm $\|\cdot\|$; let $B(H)$ denote the space of bounded linear operators on H, with the usual operator norm, also denoted by $\|\cdot\|$. A subset $\mathbb{S}$ of $B(H)$ is called proximinal in $B(H)$ if for every T in $B(H)$ there exists at least one X_0 in $\mathbb{S}$ such that $\|T - X_0\| = \inf\{\|T - X\|: X \text{ in } \mathbb{S}\}$; such an operator X_0 is called an $\mathbb{S}$-approximant of T. For each nonempty subset Λ of the complex numbers $\mathbf{C}$, denote by $N(\Lambda)$ the set of all normal operators in $B(H)$ with spectrum included in Λ. Such sets are studied by P. R. Halmos in [2], where it is shown that if Λ is closed, then $N(\Lambda)$ is norm closed and, moreover, each normal operator in $B(H)$ has an $N(\Lambda)$-approximant. If H is finite dimensional, then an easy compactness argument shows that for any closed set Λ the set $N(\Lambda)$ is proximinal in $B(H)$. If H is infinite dimensional, then for $\Lambda = [0, \infty)$ or $\Lambda =$ all real numbers, the set $N(\Lambda)$ is proximinal in $B(H)$ (see [1]), and for $\Lambda =$ the unit circle, the set $N(\Lambda)$ of unitary operators fails to be proximinal in $B(H)$ [5].

The purpose of this note is to show that if H is infinite dimensional, then neither the set of normal operators ($\Lambda = \mathbf{C}$) nor the set of orthogonal projections ($\Lambda = \{0, 1\}$) is a proximinal subset of $B(H)$; this latter result answers a question raised in [2].

1. **Normal operators.** We shall use the following definition: a nonzero vector f in H is called a maximal vector for an operator X if $\|Xf\| = \|X\| \cdot \|f\|$. It is not difficult to see that f is a maximal vector for X if and only if $X^*Xf = \|X^*X\| \cdot f$.

1.1 LEMMA. *If f is a maximal vector for X and if g is any vector such that $(f, g) = 0$, then $(Xf, Xg) = 0$.*

PROOF. $(Xf, Xg) = (X^*Xf, g) = \|X^*X\| \cdot (f, g)$.

For T in $B(H)$ write $d(T) = \inf\{\|T - N\|: N \text{ normal}\}$.

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1.2 THEOREM. *If T is an operator with a dense range such that $d(T) \leqslant \|T\|/2$ and such that the kernel of T contains a maximal vector for the adjoint of T, then T fails to have a normal approximant.*

PROOF. Assume, without loss of generality, that $\|T\| = 1$, and let f be a unit vector such that $Tf = 0$ and $\|T^*f\| = 1$. It follows, as in [3] and [4], that these properties of f imply $d(T) \geqslant \frac{1}{2}$; hence $d(T) = \frac{1}{2}$ by the hypothesis on $d(T)$. We now show that not only is there no normal operator at this distance from T, but also there is no hyponormal operator N such that $\|N - T\| \leqslant \frac{1}{2}$ (an operator N is hyponormal if $N^*N \geqslant NN^*$).

Suppose there exists a hyponormal operator N such that $\|N - T\| \leqslant \frac{1}{2}$. Then $N^*f = Tf^*/2$ because $\|N^*f\| \leqslant \|Nf\| = \|Nf - Tf\| \leqslant \frac{1}{2}$ and also $\|N^*f - T^*f\| \leqslant \frac{1}{2}$; the only vector in H satisfying both these inequalities is $T^*f/2$. Hence $\|N^*f\| = \|Nf\| = \frac{1}{2}$.

We shall now obtain a contradiction by proving $Nf = 0$. Observe that $N^*Nf = NN^*f$ since $\|Nf\| = \|N^*f\|$ and N is hyponormal. Assertion: if g is any vector such that $(f, g) = 0$, then $(Nf, Tg) = 0$. Proof:

$$\begin{aligned}
(Nf, Tg) &= (Nf, Ng) - (Nf, (N - T)g) \\
&= (Nf, Ng) - ((N - T)f, (N - T)g) \text{ because } Tf = 0 \\
&= (Nf, Ng) \text{ because } f \text{ is maximal for } N - T \text{ and Lemma 1.1 applies} \\
&= (N^*f, N^*g) \text{ since } N^*Nf = NN^*f \\
&= (T^*f/2, N^*g) \text{ since } N^*f = T^*f/2 \\
&= (T^*f/2, N^*g) - (T^*f/2, T^*g) \text{ because } f \text{ is maximal for } T^* \\
&= (T^*f/2, (N^* - T^*)g) \\
&= -((N^* - T^*)f, (N^* - T^*)g) \text{ because } T^*f/2 = -(N^* - T^*)f \\
&= 0 \text{ because } f \text{ is maximal for } N^* - T^*.
\end{aligned}$$

Thus $(Nf, Tg) = 0$ if $(f, g) = 0$. Because also $Tf = 0$, we can conclude that Nf is orthogonal to the range of T. Because the range of T is dense, this implies $Nf = 0$. Hence no such hyponormal operator N exists. This proves Theorem 1.2.

1.3 EXAMPLE. The hypotheses of Theorem 1.2 are satisfied by the operator $T =$ the adjoint of the unilateral weighted shift with weight sequence $(1, 1/2, 1/3, \ldots, 1/n, \ldots)$.

To see this, let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of H such that $T^*e_s = e_{s+1}/s$, for $s = 1, 2, \ldots$. Clearly the range of T is dense, and e_1 is a maximal vector for T^* that is in the kernel of T. To show $d(T) \leqslant \frac{1}{2}$, define the normal operators N_k for $k = 2, 3, \ldots$ by $N_k e_1 = e_k/2$, $N_k e_s = e_{s-1}/2$ for $2 \leqslant s \leqslant k$ and $N_k e_s = e_s/2$ for $s > k$. Then N_k is normal ($2N_k$ is a unitary operator) and $\|N_k - T\| \leqslant (1/2) + (1/k)$. Thus $d(T) \leqslant \frac{1}{2}$. This inequality also follows from [4, Theorem 3] since T is compact.

2. **Projections.** In this section we show that if H is infinite dimensional, then the set of orthogonal projections is not proximinal in $B(H)$. We prove, in fact, the following more general result.

2.1 THEOREM. *If Λ is a nonempty compact subset of the real line, then $N(\Lambda)$ is proximinal if and only if Λ is an interval.*

PROOF. If Λ is a closed interval, then $N(\Lambda)$ is closed in the weak operator topology and hence is proximinal (see [1, p. 956]).

Conversely, assume Λ is not an interval. Notice that if $\Lambda' = \{\alpha\lambda + \beta: \lambda \text{ in } \Lambda\}$ is the image of Λ under the affine transformation $\alpha\lambda + \beta$ for some (fixed) numbers α and β with $\alpha \neq 0$, then $N(\Lambda')$ is proximinal if and only if $N(\Lambda)$ is proximinal. Hence we assume that -1 and $+1$ are both in Λ and $|\lambda| \geqslant 1$ for all λ in Λ. Write $a = \min\Lambda$ and $b = \max\Lambda$, so that $\Lambda \subseteq [a,-1] \cup [1,b]$. Choose $c > 0$ and let $\{e_1, e_2, e_3, \ldots\}$ be an orthonormal basis for H. With respect to the decomposition span $\{e_1, e_2, e_3\} \oplus$ span $\{e_4, e_5, e_6, \ldots\}$, define A in $B(H)$ by

$$A = \begin{pmatrix} a & ic & 0 \\ ic & 0 & ic \\ 0 & ic & b \end{pmatrix} \oplus \operatorname{diag}(1, 1/2, 1/3, \ldots, 1/n, \ldots).$$

Assertion: The operator A fails to have an $N(\Lambda)$-approximant. Proof: Notice first that dist $(A, N(\Lambda)) = (1 + 2c^2)^{\frac{1}{2}}$. To see this, let N be in $N(\Lambda)$. Then

$$\begin{aligned}\|N - A\|^2 + \|N - A^*\|^2 &\geqslant \|(N - A)e_2\|^2 + \|(N - A^*)e_2\|^2 \\ &= \|(N - A)e_2\|^2 + \|(N + A)e_2\|^2 \\ &= 2(\|Ne_2\|^2 + \|Ae_2\|^2) \geqslant 2(1 + 2c^2).\end{aligned}$$

To prove the reverse inequality, define N_k in $N(\Lambda)$ for $k = 4, 5, \ldots$ by $N_k e_1 = ae_1$, $N_k e_2 = e_k$, $N_k e_3 = be_3$, $N_k e_k = e_2$ and $Ne_j = e_j$ otherwise; then $\|N_k - A\| \leqslant (1 + 2c^2)^{\frac{1}{2}} + (1/(k-3))$. Thus

$$\operatorname{dist}(A, N(\Lambda)) = (1 + 2c^2)^{\frac{1}{2}}.$$

Now suppose there exists an N in $N(\Lambda)$ such that $\|N - A\| = (1 + 2c^2)^{\frac{1}{2}}$. The arguments above show that in this case e_2 is a maximal vector for both $N - A$ and $N - A^*$ and $\|Ne_2\| = 1$. We now obtain a contradiction by proving $Ne_2 = 0$.

We show first that Ne_2 is orthogonal to $\{e_1, e_2, e_3\}$. Let N have the matrix representation (x_{st}) with respect to the orthonormal basis $\{e_1, e_2, \ldots\}$ $(s, t = 1, 2, \ldots;\ x_{st} = \bar{x}_{ts})$. Lemma 1.1 implies

$$((N - A)e_1, (N - A)e_2) = 0 = ((N - A^*)e_1, (N - A^*)e_2).$$

Hence

$$(x_{11}-a)(x_{21}+ic)+(x_{21}-ic)x_{22}+x_{31}(\bar{x}_{32}+ic)+\sum_{s=4}^{\infty}x_{s1}\bar{x}_{s2}=0$$

and

$$(x_{11}-a)(x_{21}-ic)+(x_{21}+ic)x_{22}+x_{31}(\bar{x}_{32}-ic)+\sum_{s=4}^{\infty}x_{s1}\bar{x}_{s2}=0.$$

Subtracting these two equations and dividing by $2ic$ yields $(x_{11}-a)-x_{22}+x_{31}=0$.

We can also conclude from Lemma 1.1 that

$$((N-A)e_2,(N-A)e_3)=0=((N-A^*)e_2,(N-A^*)e_3).$$

Hence

$$(\bar{x}_{21}-ic)x_{31}+x_{22}(x_{32}+ic)+(x_{32}-ic)(x_{33}-b)+\sum_{s=4}^{\infty}x_{s2}\bar{x}_{s3}=0$$

and

$$(\bar{x}_{21}+ic)x_{31}+x_{22}(x_{32}-ic)+(x_{32}+ic)(x_{33}-b)+\sum_{s=4}^{\infty}x_{s2}\bar{x}_{s3}=0.$$

Subtracting these two equations and dividing by $-2ic$ yields $(x_{33}-b)-x_{22}+x_{31}=0$.

From these observations we can conclude that $x_{11}-a=x_{22}-x_{31}=x_{33}-b$; thus $x_{33}-x_{11}=b-a$. Hence x_{11} and x_{33} are two points in the numerical range of N that are as far apart as any two points in the numerical range can be (since the spectrum of N is included in $[a,b]$); this implies that $x_{11}=a$ and $x_{33}=b$ and consequently $x_{22}=x_{31}$. Since extreme points in the numerical range of N must be eigenvalues, we conclude also that $Ne_1=ae_1$ and $Ne_3=be_3$.

Since $Ne_1=ae_1$, we have $x_{12}=\bar{x}_{21}=0$ and $x_{22}=x_{31}=0$; since $Ne_3=be_3$, we have $x_{32}=\bar{x}_{23}=0$. Thus Ne_2 is orthogonal to $\{e_1,e_2,e_3\}$.

It now remains to show $x_{t2}=0$ for $t=4,5,\ldots$. Since $\|Ne_2\|=1$ and N is bounded below by 1, it follows that $N^2e_2=e_2$; thus $(Ne_2,Ne_t)=0$ for $t\neq 2$. Since $x_{12}=x_{22}=x_{32}=0$, it follows that $(Ne_2,Ne_t)=\sum_{s=4}^{\infty}x_{s2}\bar{x}_{st}=0$.

Because e_2 is maximal for $N-A$, it also follows that $((N-A)e_2,(N-A)e_t)=0$. This implies for $t\geqslant 4$ that

$$0=((N-A)e_2,(N-A)e_t)=\left(\sum_{s=4}^{\infty}x_{s2}\bar{x}_{st}\right)-\frac{x_{t2}}{t-3}$$

because $Ae_t=e_t/(t-3)$ and $x_{1t}=x_{3t}=x_{22}=0$. Using the fact that $\sum_{s=4}^{\infty}x_{s2}\bar{x}_{st}=0$, we can thus conclude that $x_{t2}=0$ for $t=4,5,\ldots$. Hence $Ne_2=0$. This proves Theorem 2.1.

The preceding results contribute to a characterization of those sets Λ for which $N(\Lambda)$ is proximinal in $B(H)$. The only sets Λ for which it is known that $N(\Lambda)$ is proximinal are $\Lambda =$ one point, $\Lambda = [0, 1]$, $\Lambda = [0, \infty)$, $\Lambda = (-\infty, \infty)$, or an affine translation of one of these intervals; these sets $N(\Lambda)$ are all closed in the weak operator topology and are thus proximinal. It would be interesting to determine whether there are any other sets Λ for which $N(\Lambda)$ is proximinal.

It is not difficult to see that if $N(\Lambda)$ is proximinal, then Λ must have empty interior. Proof: If Λ includes an open set, then (by using an affine transformation $\alpha\lambda + \beta$ if necessary) we can assume that Λ includes the circle of radius $\frac{1}{2}$ with center at the origin. Hence the operators N_k in the example in §1 are in $N(\Lambda)$, and the weighted shift in that example fails to have an $N(\Lambda)$-approximant.

Thus int $(\Lambda) = \varnothing$ is a necessary condition for $N(\Lambda)$ to be proximinal. This condition is satisfied, of course, by subsets of the real line, but even in this special case the problem is not solved. In particular, for $\Lambda = (-\infty, -1] \cup [1, \infty)$ it is apparently not known whether $N(\Lambda)$ is proximinal.

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