PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, November 1976

AN EXISTENCE THEOREM FOR BOUNDARY VALUE PROBLEMS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Let f be continuous on $(a,b) \times R^n$ and suppose solutions of initial value problems for $y^{(n)} = f(t,y,\ldots,y^{(n-1)})$ exist on (a,b). Relaxing the assumption that solutions of initial value problems are unique, global existence of solutions of the boundary value problem

$$y^{(n)} = f(t,y,\ldots,y^{(n-1)}), \quad y(t_i) = \alpha_i \quad \text{for } 1 \leqslant i \leqslant n,$$

is established assuming uniqueness of solutions of these problems and a compactness property of solutions of the differential equation.

Questions concerning the global existence of solutions of boundary value problems of the type

$$y^{(n)} = f(t,y,\ldots,y^{(n-1)}), \tag{1}$$

$$y(t_i) = \alpha_i, \qquad i = 1, 2, \ldots, n, \tag{2}$$

have received considerable attention recently. The first existence theorem of this nature was established by Lasota and Opial [7] for $n = 2$. Jackson [3] and Jackson and Schrader [4] proved a similar theorem for $n = 2$ and $n = 3$, respectively, assuming only the uniqueness of solutions of such boundary value problems and the extendibility of solutions of initial value problems. More recently, Hartman [2], Klaasen [5] and Jackson [8] obtained similar results for $n > 3$ assuming additionally a compactness property of solutions to (1) and the uniqueness of solutions of initial value problems. It is the purpose of this note to prove this result without the uniqueness required on solutions of initial value problems for (1). The proof of this result, Theorem 3, makes use of the topological properties of a set-valued mapping which in the aforementioned papers was a point mapping tractable by Brouwer's theorem on invariance of domain. The extension of these properties to set-valued mappings is accomplished in [6]. The paper concludes with an example of an equation which possesses the global existence and uniqueness property for

Received by the editors May 1, 1975 and, in revised form, March 19, 1976.
AMS (MOS) subject classifications (1970). Primary 34B15.
Key words and phrases. Ordinary differential equations, boundary value problems.

solutions of boundary value problems but for which solutions of initial value problems are not unique.

Suppose $f \in C[(a,b) \times R^n]$. The following conditions are requirements for the global existence of solutions of boundary value problems (1), (2).

(A) All solutions of all initial value problems for (1) exist on (a, b).

(B) For each $\alpha_1, \ldots, \alpha_n$ and $t_1, \ldots, t_n$ with $a < t_1 < \cdots < t_n < b$, if $u(t)$ and $v(t)$ are solutions of (1), (2), then $u(t) = v(t)$ on $[t_1, t_n]$.

(C) If $\{u_j(t)\}$ is a monotone bounded sequence of solutions of (1) on $[c,d] \subset (a,b)$, then $\lim_j u_j(t)$ is a solution of (1) on $[c,d]$.

Condition (A) has the effect that any solution of (1) on any subinterval can be extended to be a solution on (a, b). Theorem 1 and Lemma 2 are necessary to prove Theorem 3. A proof of Theorem 1 appears in [6] and will not be proved here.

THEOREM 1. *Suppose equation* (1) *is such that* (A) *and* (B) *are satisfied and suppose* $y(t)$ *is a solution of* (1) *on* (a, b). *For any* $t_1, \ldots, t_n$ *such that* $a < t_1 < \cdots < t_n < b$ *and any* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if* $|y(t_i) - \alpha_i| < \delta$ *for* $1 \leqslant i \leqslant n$, *then the boundary value problem* (1), (2) *has a solution* $u(t)$ *and* $|u^{(i)}(t) - y^{(i)}(t)| < \varepsilon$ *on* $[t_1, t_n]$ *for* $0 \leqslant i \leqslant n-1$.

LEMMA 2. *Suppose equation* (1) *is such that* (A) *and* (B) *are satisfied. If* $a < t_1 < \cdots < t_n < b$ *and* $u(t)$ *and* $v(t)$ *are solutions of* (1) *on* (a, b) *satisfying* $u(t_i) = v(t_i)$ *for* $1 \leqslant i \leqslant n-1$, *then* $u(t) - v(t)$ *changes sign at each* t_i, $1 \leqslant i \leqslant n-1$.

PROOF. Choose t_0 and t_n such that $a < t_0 < t_1 < \cdots < t_{n-1} < t_n < b$. Suppose to the contrary that for some k, $1 \leqslant k \leqslant n-1$, that $u(t) - v(t) > 0$ in a deleted neighborhood of t_k. Condition (B) implies that $u(t) - v(t) > 0$ on (t_{k-1}, t_{k+1}) except at t_k. Define $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ as follows:

$$\varepsilon_1 = \sup\{u(t) - v(t) \mid t_{k-1} \leqslant t \leqslant t_k\},$$

$$\varepsilon_2 = \sup\{u(t) - v(t) \mid t_k \leqslant t \leqslant t_{k+1}\}.$$

If $k > 1$, Theorem 1 implies that there exists a solution $w(t)$ of (1) on (a,b) such that for some $\varepsilon > 0$, $w(t_i) = v(t_i)$, $1 \leqslant i \leqslant n$, $i \neq k$, and $w(t_k) = v(t_k) + \varepsilon$. Moreover, $|w(t) - v(t)| < \varepsilon_1 \wedge \varepsilon_2$ on $[t_0, t_n]$. From this it follows that $w(t)$ and $u(t)$ have n zeros in common on $[t_1, t_n]$ but are distinct solutions. This violates (B) and hence $u(t) - v(t)$ changes sign at $t = t_k$, $k > 1$. If $k = 1$ then define w as above at the points $t_0, t_1, \ldots, t_{n-1}$ and the same argument shows that $u(t) - v(t)$ changes sign at t_1.

THEOREM 3. *Suppose equation* (1) *is such that* (A), (B) *and* (C) *are satisfied. Then for each* $\alpha_1, \ldots, \alpha_n$ *and each* $t_1, \ldots, t_n$ *such that* $a < t_1 < \cdots < t_n < b$ *the boundary value problem* (1), (2) *has a unique solution.*

PROOF. It suffices to show that given $a < t_1 < \cdots < t_n < b$, a solution $y(t)$ of (1) on (a,b), and some k satisfying $1 \leqslant k \leqslant n$, that the set S of all $u(t_k)$

such that $u(t)$ is a solution of (1) on (a, b) and $u(t_i) = y(t_i)$ for $1 \leqslant i \leqslant n$, $i \neq k$, is the set of all real numbers. Theorem 1 implies that S is open so it suffices to show that S is closed. Suppose $\beta_j \in S$ for $j = 1, 2, \ldots$ and $\lim_j \beta_j = \beta_0$. Without loss of generality suppose $\{\beta_j\}$ is an increasing sequence. Let t_0 and t_{n+1} be chosen so that $a < t_0 < t_1 < \cdots < t_n < t_{n+1} < b$. Let $u_j(t)$ be the solution of (1) such that $u_j(t_i) = y(t_i)$ for $1 \leqslant i \leqslant n$, $i \neq k$, and $u_j(t_k) = \beta_j$ for $j = 1, 2, 3, \ldots$. If there is a subinterval $[c, d]$ of $[t_0, t_{n+1}]$ on which $\{u_j(t)\}$ is bounded, then by Lemma 1 $[c, d]$ possesses a subinterval on which $\{u_j(t)\}$ is both monotone and bounded and, hence, property (C) implies that $\lim_j u_j(t) \equiv u_0(t)$ is a solution on that interval. Theorem 1 implies that for any s_0 in that interval $\lim u_j^{(i)}(s_0) = u_0^{(i)}(s_0)$ for $0 \leqslant i \leqslant n-1$. Kampke's theorem [1, p. 14] implies that a subsequence of u_j converges uniformly on compact subsets of (a, b), in particular, on $[t_0, t_{n+1}]$, to a solution of (1). That solution $v(t)$ must satisfy $v(t_i) = y(t_i)$ for $1 \leqslant i \leqslant n$, $i \neq k$, and $v(t_k) = \beta_0$. Thus $\beta_0 \in S$. On the other hand, suppose $\{u_j(t)\}$ is not bounded on any subinterval of $[t_1, t_{n+1}]$. For each i, $1 \leqslant i \leqslant n$, let $\eta_i = (t_i - \delta_i, t_i + \delta_i)$ where $t_0, t_{n+1} \notin \eta_i$ and $\eta_i \cap \eta_j = \varnothing$ for $j \neq i$. Since $u_j(t) - y(t)$ changes signs at t_i for $1 \leqslant i \leqslant n$, $i \neq k$, if $w(t)$ is any solution of (1) on (a, b) such that $w(t_k) = \beta_0$, then for j sufficiently large, $u_j(t) - w(t)$ has a zero in each of η_i for $1 \leqslant i \leqslant n$, $i \neq k$. Moreover, since $w(t_k) > u_j(t_k) > u_{j-1}(t_k)$, $j \geqslant 1$, for j sufficiently large $w(t) - u_j(t)$ must have two zeros in η_k. But then $w(t) - u_j(t)$ has $n + 1$ zeros for j sufficiently large which violates (B). Thus the second alternative is eliminated and the theorem is proved.

As an example of a differential equation for which conditions (A), (B) and (C) hold, and hence global existence, but for which initial value problems do not have unique solutions, consider the equation

$$y^{\mathrm{IV}} = (y'')^{1/3} \tag{3}$$

and any interval (a, b). It is important to relate (3) with the equation

$$u'' = u^{1/3} \tag{4}$$

for which global existence and uniqueness of solutions of associated boundary value problems $u(t_1) = \alpha_1$, $u(t_2) = \alpha_2$ is known, but initial value problems do not have unique solutions even though all such solutions extend to the entire interval.

The following relationship between (3) and (4) is useful for establishing the above claims.

If y is a solution of (3) on an interval I then $u = y''$ is a solution of (4) on I, and, conversely, if u is a solution of (4) on I, $x_0 \in I$ and $\alpha, \beta \in R$ then

$$y(t) = \int_{t_0}^{t} \int_{t_0}^{s} u(x)\, dx\, ds + \alpha t + \beta$$

is a solution of (3) on I.

Condition (A) follows from this relationship and the fact that solutions of (4) exist on (a, b).

To prove condition (B) suppose y and z are solutions of (3) on $[t_1, t_4]$

$\subset (a,b)$ and $y(t_i) = z(t_i)$ for $t_1 < t_2 < t_3 < t_4$. Then by repeated use of the Mean Value Theorem there are τ_1, τ_2, τ_3 and σ_1, σ_2 such that $t_1 < \tau_1 < t_2 < \tau_2 < t_3 < \tau_3 < t_4$ and $\tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \tau_3$ for which $y'(\tau_i) = z'(\tau_i)$, $i = 1$, 2, 3, and $y''(\sigma_i) = z''(\sigma_i)$, $i = 1, 2$. Observing that y'' and z'' are solutions of (4) for which uniqueness of 2 point boundary value problems is easy to establish, we conclude that $y''(t) = z''(t)$ on $[\sigma_1, \sigma_2]$. In fact whenever $y''(s_1) = z''(s_1)$ and $y''(s_2) = z''(s_2)$, $s_1 < s_2$, then $y''(t) = z''(t)$ on $[s_1, s_2]$. Let $[c,d]$ be the largest such interval in (a, b). Knowing that $\tau_2 \in [c,d]$ we conclude that $y'(t) = z'(t)$ on $[c,d]$. By repeated applications of the Mean Value Theorem, $\tau_3 > d$ and $\tau_1 < c$ can be eliminated. Thus $t_2 \in [c,d]$ and hence one can argue that $y(t) = z(t)$ on $[c,d]$. One more application of the Mean Value Theorem yields t_1 and t_4 in $[c,d]$ and uniqueness is established.

Finally an argument for compactness is presented. Suppose $\{y_n\}$ is a monotone increasing sequence of solutions of (3) which converge pointwise to a bounded function y on $[\alpha,\beta]$. Since y''_n is a solution of (4), and hence has at most one zero, we can assume, by shortening the interval $[\alpha,\beta]$ and taking subsequences if necessary, that y''_n is of one sign for each n. To be specific suppose $y''_n(x) \geqslant 0$ for all $n \geqslant 1$ and $\alpha \leqslant x \leqslant \beta$. Thus $y_n^{\mathrm{IV}} \geqslant 0$ on $[\alpha,\beta]$ for all $n \geqslant 1$. Hence for nonlinear solutions y''_n has exactly one minimum on $[\alpha,\beta]$. Again by taking subsequences and shortening $[\alpha,\beta]$ if necessary, we can assume that for each n, y''_n is monotone on $[\alpha,\beta]$. To be specific suppose y''_n is increasing on $[\alpha,\beta]$ for each $n \geqslant 1$. But since $\{y_n\}$ are bounded on $[\alpha,\beta]$ one can conclude that given $\varepsilon > 0$, there exists β_n, $\beta - \varepsilon < \beta_n < \beta$, and $M > 0$ such that $|y''_n(\beta_n)| \leqslant M$ for $n \geqslant 1$. Thus $\{y''_n\}$ is bounded on $[\alpha, \beta - \varepsilon]$ and so also is $\{y_n^{\mathrm{IV}}\}$. Applications of the Ascoli-Arzela Theorem yield that $\lim_{n\to\infty} y_n(x) = y(x)$ is a solution of (3) on some subinterval of $[\alpha,\beta]$. By appealing to Kampke's Theorem as in the proof of Theorem 3 we conclude that y is a solution of (3) on the entire interval.

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