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A MEAN ERGODIC THEOREM FOR FAMILIES OF CONTRACTIONS IN HILBERT SPACE

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ABSTRACT. Let G be an LCA group and h a Hilbert space, and let $T(g)$ be a function on G into the contractions on H. Let $\{\sigma_n\}$ be a sequence of probability measures on G. Under suitable conditions on $T(g)$ and the sequence $\{\sigma_n\}$ we prove the strong convergence of the sequence $T_n = \int T(g)\sigma_n(dg)$. In certain cases we identify the limiting operator.

Let G be an LCA group and $\mathcal{B}(G)$ the Borel sets of G. $\mathcal{Q}$ will denote the class of functions $l(\gamma)$ defined on $\hat{G}$ which are Fourier transforms of a probability measure defined on $\mathcal{B}(G)$. If $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence of such probability measures, with $\lim_n \sigma_n(\gamma) = l(\gamma)$ for $\forall \gamma \in \hat{G}$, we shall call the sequence $l(\gamma)$-*amenable.*

A set $S \in \mathcal{B}(G)$ is called *amenable* if $\exists$ a nonnegative Borel measure μ on $\mathcal{B}(G)$ and a sequence $\{S_n\}$ of Borel subsets of S such that

(i) $\mu(B + g) = \mu(B)$ for every $B \in \mathcal{B}(G)$ and $g \in S$,

(ii) $0 < \mu(S_n) < n = 1, 2, \ldots,$

(iii) $\lim_{n\to\infty} \mu[(S_n + g) \cap S_n]/\mu(S_n) = 1$ for $\forall g \in S$.

If

$$l_S(\gamma) = \begin{cases} 1 & \text{if } \langle g, \gamma \rangle = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } \forall g \in S,$$

then it is clear that $l_S \in \mathcal{Q}$ and, in fact, if $\sigma_n(\cdot) = \mu[S_n \cap (\cdot)]/\mu[S_n]$, then $\lim_n \hat{\sigma}_n(\gamma) = l_S(\gamma)$ for $\forall \gamma \in \hat{G}$.

Now let $T(g)$ be a function defined on G whose values are contractions on a Hilbert space. We shall assume that $T(g)$ is positive definite. For a definition of a positive definite valued function, see Foiaş and Nagy [2, p. 25].

With these definitions we can state our main result.

THEOREM . *Let G be an LCA group and H a Hilbert space. Let $T(g)$ be a positive definite function from G to the bounded operators on H such that $T(e)$ is the identity operator and $T(g)$ is weakly continuous. Then*

(i) *$T(g)$ is strongly continuous.*

(ii) *For a probability measure σ on G the integral $\int_G T(g)\sigma(dg)$ exists strongly and is a contraction operator on H.*

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(iii) *With every function* $l(\gamma) \in \mathcal{Q}$ *is associated a contraction operator* L^l *on* H *so that if* $\{\sigma_n\}_{n=1}^{\infty}$ *is an* $l(\gamma)$*-sequence of probability measures on* G, *then* $L^l = \text{strong-lim}_{n\to\infty} \int_G T(g)\sigma_n(dg)$.

(iv) *If* S *is an amenable set and* $T(g + g') = T(g) \cdot T(g')$ *for* $g, g' \in S$, *and if* $\{\sigma_n\}_{n=1}^{\infty}$ *is an* $l_S(\gamma)$*-sequence of probability measures on* G, *then* $\text{strong-lim}_{n\to\infty} \int_G T(g)\sigma_n(dg) = P_{\tilde{S}}$ *where* $P_{\tilde{S}}$ *is the orthogonal projection onto the closed linear manifold* $\tilde{S} = \{h \in H \mid T(g)h = h \text{ for } \forall g \in S\}$.

OUTLINE OF THE PROOF. The use of the *dilation theorem* in [2, p. 25] will be indicated by (*).

(i) as defined in (*) we have

$$\|[T(g) - T(g')]h\|_H^2 = ([U(g) - U(g')]h, [T(g) - T(g')]h)_K.$$

Now use Schwartz's inequality and (*).

(ii) follows from (i).

(iii) Let $E(\cdot)$ be the resolution of the identity corresponding to $U(g)$ (as defined in (*)). Define for $l \in \mathcal{Q}$: $N^l = \int_{\hat{G}} l(\gamma)E(d\gamma)$; clearly N^l is a contraction on K. Now let $L^l = P_H N^l$; P_H is the orthogonal projection onto H. If $\{\sigma_n\}_{n=1}^{\infty}$ is an $l(\gamma)$-sequence and $h \in H$, then

$$\begin{aligned}
&\left\| \int_G T(g)h\sigma_n(dg) - P_H N^l h \right\|^2 \\
&\qquad = \left(\int_G U(g)h\sigma_n(dg) - N^l h, \int_G T(g)h\sigma_n(dg) - P_H N^l h \right)_K \\
&\qquad \leqslant \left\| \int_G U(g)h\sigma_n(dg) - N^l h \right\|_K^2 \cdot 2 \cdot \|h\|_H.
\end{aligned}$$

Now use the same technique as in the proof of (*) to finish the proof.

(iv) We have to show that $L^{l_S} = P_{\tilde{S}}$ (using (iii)). Since S is amenable the sequence of probability measures $\mu_n(\cdot) = \mu(S_n \cap (\cdot))/\mu(S_n)$ is an l_S-sequence. Now let $h \in H$ and $f \in \tilde{S}$; then

$$\begin{aligned}
(h - L^{l_S}h, f)_H &= \left(h - \lim_{n\to\infty} \int_G T(g)h\mu_n(dg), f \right)_H \\
&= (h,f)_H - \lim_{n\to\infty} \frac{1}{\mu(S_n)} \int_{S_n} (h, T^*(g)f)_H \mu(dg) = 0.
\end{aligned}$$

Therefore $h - L^{l_S}h \in (\tilde{S})^{\perp}$. Now let $g' \in S$; then

$$\begin{aligned}
T(g')L^{l_S}h &= T(g') \text{ strong-}\lim_{n\to\infty} \int_G T(g)h\mu_n(dg) \\
&= \lim_{n\to\infty} \frac{1}{\mu(S_n)} \int_{S_n} T(g')T(g)h\mu(dg).
\end{aligned}$$

Therefore

$$L^{ls}h - T(g')L^{ls}h$$

$$= \lim_{n\to\infty}\left[\frac{1}{\mu(S_n)}\int_{S_n} T(g)h\mu(dg) - \frac{1}{\mu(S_n)}\int_{S_n} T(g'+g)h\mu(dg)\right]$$

$$= \lim_{n\to\infty}\frac{1}{\mu(S_n)}\int_{S_n\triangle(S_n+g')} T(g)h\mu(dg).$$

Therefore,

$$\|L^{ls}h - T(g')L^{ls}h\|_H = \lim_{n\to\infty}\left\|\frac{1}{\mu(S_n)}\int_{S_n\triangle(S_n+g')} T(g)h\mu(dg)\right\|_H$$

$$\leqslant \lim_{n\to\infty}\frac{\mu[S_n\triangle(S_n+g')]}{\mu(S_n)}\|h\|_H = 0.$$

Therefore $L^{ls}h \in \tilde{S}$ which together with $h - L^{ls}h \in (\tilde{S})^{\perp}$ implies $L^{ls} = P_{\tilde{S}}$. Q.E.D.

References

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