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ON THE ZEROS OF GENERALIZED AXIALLY SYMMETRIC POTENTIALS*

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ABSTRACT. Generalized axially symmetric potentials may be expanded as Fourier-Jacobi series in terms of the complete system $r^k C_k^{n/2-1}(\cos\theta)$ on axisymmetric regions $\Omega \subset E^n$ ($n \geqslant 3$) about the origin. The values of these potentials are characterized by the nonnegativity of sequences of determinants drawn from the Fourier coefficients in a manner analogous to the characterization of the values of analytic functions of one complex variable by the theorems of Carathéodory-Toeplitz and Schur.

1. **Introduction.** The singularities of generalized axially symmetric potentials (GASP) ψ are characterized from the coefficients of their Fourier-Jacobi series expansions by S. Bergman [1], [2] and R. P. Gilbert [3], [4] who found a definitive characterization by employing function theoretic means to develop a theory mirroring the classical work of Hadamard, Mandelbrojt and Fabry on the singularities of analytic functions f of one complex variable. Cauchy's classical estimate [5, p. 123] on the zeros of a polynomial, along with the theorems of Carathéodory-Toeplitz [12, p. 157] and Schur [12, p. 159] bound the zeros of f relative to convex sets from its coefficients; the possibility arises of developing a parallel theory for the values of ψ.

An extension of Cauchy's estimate applying to the zeros of harmonic polynomials in E^n was found by Morris Marden [8]. We shall illustrate how the reasoning in [8] which is based on methods found in Professor Marden's early work [6], [7] can be used to locate the zeros of GASP. Moreover, it is apparent that this reasoning plays a role in bounding the zeros of ψ analogous to the role played by the reasoning of Hadamard in the multiplication of singularities theorem upon which Gilbert's "envelope method" [1], [2] is based. *Remark*: Investigations [9]–[11] subsequent to [8] employ the A_n operator [4, p. 168] to geometrically establish equality of known sets of excluded values of the A_n associates and corresponding sets of excluded values of GASP.

2. **On the zeros.** In axisymmetric regions $\Omega \subset E^n$ [8] about the origin, generalized axially symmetric potentials ψ_n may be developed as a series [4, p. 168]

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(1) $$\psi_n(r, \theta) = \psi_n(\mathbf{X}) = \sum_{k=0}^{\infty} B(n-2, k+1) a_k r^k C_k^{n-1/2}(\cos\theta)$$

in terms of the polar coordinates (r, θ) which depend on the cylindrical coordinates $x = x_1$, $\rho = (x_2^2 + \cdots + x_n^2)^{1/2}$ by $x = r\cos\theta$, $\rho = r\sin\theta$. Because of symmetry, ψ_n assumes constant values of hypercircles $\mathbf{X} = (x, \rho)$ in Ω.

For each ψ_n with domain Ω, there is a unique associated function f,

(2) $$f(\xi) = \sum_{k=0}^{\infty} a_k \xi^k,$$

analytic on the corresponding axiconvex domain $\omega \subset \mathbf{C}$ [8] whose A_n transform [4] is defined by

(3) $$\psi_n(\mathbf{X}) = \alpha_n \int_L f(\sigma) d\nu_n(\zeta),$$

$\sigma = x + \rho/2(\zeta + \zeta^{-1})$, $\alpha_n = \Gamma(n-2)/(4i)^{n-2}\Gamma(n/2 - 1)$ with nonnegative measure

(4) $$d\nu_n(\zeta) = (\zeta - \zeta^{-1})^{n-3} d\zeta/\zeta$$

for ζ on the contour $L \equiv \{\zeta = e^{it} | 0 \leqslant t \leqslant \pi\}$. (The Fourier-Legendre expansion for $n = 3$ follows from the Bergman-Whittaker operator.) From the development of ψ_n in (1) we defined the matrix

(5) $$M_k = \begin{pmatrix} b_0, a_1, \ldots, a_k \\ 0,\ b_0, \ldots, a_{k-1} \\ \ldots\ldots\ldots\ldots\ldots \\ 0, 0, 0, \ldots, b_0 \end{pmatrix}$$

its conjugate transpose $\overline{M}_k'$ ($b_0 \equiv a_0 - \alpha$, $\alpha \in \mathbf{C}$) and the determinants

(6) $$\sigma_k(\alpha) = \sigma_k(a_0, \ldots, a_k, \alpha) = \det\left(M_k + \overline{M}_k'\right)$$

and

(7) $$\gamma_k(\alpha) = \gamma_k(a_0, \ldots, a_k, \alpha) = \det\begin{pmatrix} M_k & I_k \\ I_k & \overline{M}_k' \end{pmatrix}$$

where I_k is the $(k+1)$ by $(k+1)$ identity matrix. Initially the values of ψ_n relative to the hypersphere S_n with unit radius about the origin are considered in

THEOREM 1. *Let the generalized axially symmetric potential ψ_n be expanded as in* (1). *Then ψ_n is regular in the hypersphere S_n and on each hypercircle* $\mathbf{X}$ *in* S_n,

(8) $$\psi_n(\mathbf{X}) \neq \alpha + \eta, \quad \forall\ \mathrm{Re}(\eta) < 0, \ (a_0 - \alpha \textit{ real})$$

or

(9) $$\psi_n(\mathbf{X}) \neq \eta, \quad \forall |\eta - \alpha| > 1$$

for η complex according to whether

(8a) $$\sigma_0(\alpha) > 0, \sigma_1(\alpha) > 0, \ldots, \sigma_k(\alpha) > 0, \ldots$$

or

(9a) $$\gamma_0(\alpha) > 0, \gamma_1(\alpha) > 0, \ldots, \gamma_k(\alpha) > 0, \ldots.$$

In particular, if $\alpha > 0$ *and* (8a) *or* $|\eta| > 1$ *and* (9a), *then* ψ_n *has no zeros in* S_n.

PROOF. When (8a) or (9a) are valid, the associate f given by (2) is regular in the disk $D = |\xi| < 1$ so that its A_n transform represents ψ_n in S_n. When (8a) is satisfied, the Carathéodory-Toeplitz theorem guarantees that $\operatorname{Re}[f(\sigma) - \alpha] \geqslant 0$ when $|\sigma| < 1$ which is the case for all $x^2 + \rho^2 < 1$ and $\zeta \in L$. Moreover, for $\zeta \in L$, the measure $d\nu_n(\zeta)$ is nonnegative. Consequently, the integral $\psi_n - \alpha = A_n(f - \alpha)$, viewed as the uniform limit of a sum of vectors, each terminating in the right half plane P, is in P (see [6], [7]). This establishes (8).

In (9a), Schur's theorem guarantees the bound $|f(\sigma) - \alpha| \leqslant 1$, $x^2 + \rho^2 < 1$, $\zeta \in L$. The measure is nonnegative and the operator is normalized according to $A_n(1) = 1$. Hence $|\psi_n(\mathbf{X}) - \alpha| \leqslant A_n(1)$. *Remark*: Due to Fatou's theorem [12, p. 146], $\alpha > 0$ is a sufficient condition for the existence of the radial limits of ψ_n relative to S_n, with the possible exception of a set of singular circles whose intersection with a meridian plane has one dimensional measure zero.

THEOREM 2. *If for some constant* α, *the sequence*

(10) $$\gamma_0(\alpha) > 0, \ldots, \gamma_k(\alpha) > 0, \quad \gamma_{k+1}(\alpha) = \cdots = 0,$$

then ψ_n *is a linear combination of Newtonian potentials whose mass is distributed over* $k + 1$ *singular circles exterior to* S_n *such that* ψ_n *has no zeros in the double cone*

(11) $$0 < \rho < \pm(R - x)\tan(\pi/4(k + 1)).$$

The distance R *from the origin to the most remote* (*finite*) *singular circle is determined by the Gilbert-Hadamard theorem. Moreover, if* ψ_n *vanishes on the sphere centre at* $x = x_0$ *and radius* $\rho = \rho_0$ *with no singular circles in the concentric sphere of radius* $\rho_0\csc(\pi/2(k + 1))$ *about* S_n, *the associate vanishes in the disk* $|\xi - x_0| \leqslant \rho_0\csc(\pi/2(k + 1))$.

PROOF. Because of (10), the associate f is a rational function with $(k + 1)$ distinct zeros and $(k + 1)$ poles which are symmetric in $|\xi| = 1$ [12, p. 159]. Application of the A_n operator to the partial fraction expansion of f establishes the first part. To verify that ψ_n has no zeros in the cones (11), we reason as in [8] bearing in mind the location of the zeros and poles of the associate. To establish the converse relation on the effect of the zeros of ψ on those of f, apply Marden's mean value theorem [5, p. 11] in a manner similar to its use in [9].

Relating the values of ψ_n on axisymmetric sets to the values assumed in convex sets suggests a lengthy statement on the Fourier coefficients. For brevity, we consider the version in

THEOREM 3. *The generalized axially symmetric potential* ψ_n *as in* (1) *has the axisymmetric set* $\Omega \subset E^n$ *for its domain where the analytic function*

(12) $$g(\xi) = \sum_{k=0}^{\infty} g_k \xi^k / k!$$

maps the unit disk onto the corresponding axiconvex set $\omega \subset \mathbf{C}$. *The constants* β_k *are defined by*

(13) $$\beta_k = \sum_{j=0}^{k} a_j j!/k! \sum P_k^{(j)} g_1^{m_1} \dots g_k^{m_1},$$

(14) $$P_k^{(j)} = k!/(1!)^{m_1} \dots (k!)^{m_k}(m_k!).$$

The summations are over $m_1 + 2m_2 + \cdots + km_k = k$, *and* $m_1 + \cdots + m_k = j$. *If either sequence,*

(15) $$\sigma_0(\beta_0, \alpha) > 0, \dots, \sigma_k(\beta_0, \dots, \beta_k, \alpha) > 0, \dots$$

or

(16) $$\gamma_0(\beta_0, \alpha) > 0, \dots, \gamma_k(\beta_0, \dots, \beta_k, \alpha) > 0, \dots,$$

then on each hypercircle $\mathbf{X} \in \Omega$, ψ_n *satisfies the corresponding inequality* (8) *or* (9).

Proof. We write $\tilde{f} = f \circ g$. Then (15) or (16) imply that f maps D onto the respective convex sets D or $\mathrm{Re}(\xi) \geqslant 0$. Consequently, $f: \omega \to D$ or $\mathrm{Re}(\xi) \geqslant 0$ so that reasoning as in Theorem 1 completes the proof.

3. **A generalization to $\Delta_3\Phi + F(r^2) = 0$.** The purpose of this closing remark is to indicate a broader range of application of [6], [7] by referring to work of S. Bergman [2] who expands axially symmetric solutions of

(17) $$L\Phi \equiv \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + F(r^2)\Phi = 0$$

(F is entire) in series of generalized Bessel functions J_n^* [2, p. 435].

Theorem 4. *If*

(18) $$\Phi(\mathbf{X}) = \sum_{k=0}^{\infty} \frac{\Gamma(1/2)A_k}{\Gamma(k+3/2)} \frac{J_k^*(r)}{J_k^*(1)} r^k P_k(\cos\theta)$$

is the development of a solution of $L\Phi \equiv 0$ *regular in the unit sphere* S_3 *and there is a constant* $|\alpha| > 1$ *for which*

(19) $$\gamma_0(A_0, \alpha) > 0, \gamma_1(A_0, A_1, \alpha) > 0, \dots, \gamma_k(A_0, \dots, A_k, \alpha) > 0,$$

then on each circle $\mathbf{X} \in S_3$,

(20) $$\Phi(\mathbf{X}) \neq 0.$$

Proof. The proof relies on the operators $\Phi = D_0(H)$ [2, p. 435] and $H = B_3(f)$. Because of (19) and (essentially) Theorem 1, $|H(\mathbf{X}) - \alpha| < 1$, $|\mathbf{X}| < 1$. Therefore, since the generating function of D_0 is real, we may conclude from [7] that $|\Phi(\mathbf{X}) - \alpha| < 1$.

Remark. More general cases can be treated using the *Method of Ascent* (see Gilbert [3]).

REFERENCES

1. S. Bergman, *Integral operators in the theory of linear partial differential equations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Heft 23, Springer-Verlag, Berlin and New York, 1961. MR **25** #5277.

2. ———, *Classes of solutions of linear partial differential equations in three variables*, Duke Math. J. **13** (1946), 419–458. MR **8**, 274.

3. R. P. Gilbert, *Constructive methods for elliptic equations*, Lecture Notes in Math., vol. 365, Springer-Verlag, Berlin and New York, 1974.

4. ———, *Function theoretic methods in partial differential equations*, Math. in Science and Engineering, Vol. 54, Academic Press, New York, 1969. MR **39** #3127.

5. M. Marden, *Geometry of polynomials*, 2nd ed., Math. Surveys, No. 3, Amer. Math. Soc., Providence, R. I., 1966. MR **37** #1562.

6. ———, *A generalization of Weierstrass' and Fekete's mean-value theorems*, Bull. Amer. Math. Soc. **38** (1932), 434–441.

7. ———, *Further mean-value theorems*, Bull. Amer. Math. Soc. **39** (1933), 750–754.

8. ———, *Value distribution of harmonic polynomials in several real variables*, Trans. Amer. Math. Soc. **159** (1971), 137–154. MR **43** #5046.

9. P. A. McCoy, *Value distribution of axisymmetric potentials*, Amer. J. Math. **95** (1973), 419–428. MR **48** #6450.

10. ———, *Value distribution of linear combinations of axisymmetric harmonic polynomials and their derivatives*, Pacific J. Math. **48** (1973), 441–450.

11. ———, *Generalized axisymmetric potentials*, J. Approximation Theory **15** (1975), 256–266.

12. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1958. MR **22** #5712.

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