

INERTIAL COEFFICIENT RINGS AND THE IDEMPOTENT LIFTING PROPERTY¹

ELLEN E. KIRKMAN

ABSTRACT. A commutative ring R with identity is called an *inertial coefficient ring* if every finitely generated R -algebra A with A/N separable over R contains a separable R -subalgebra S of A such that $A = S + N$, where N is the Jacobson radical of A . We say A has the *idempotent lifting property* if every idempotent in A/N is the image of an idempotent in A . Our main theorem is that any finitely generated algebra over an inertial coefficient ring has the idempotent lifting property.

All rings contain an identity; all subrings contain the identity of the overring; all homomorphisms preserve the identity. Throughout R denotes a commutative ring and A an R -algebra which is finitely generated as an R -module. The Jacobson radical of a ring B is denoted $\text{rad}(B)$ and throughout $\text{rad}(A) = N$. A separable R -subalgebra S of A such that $A = S + N$ is called an *inertial subalgebra*. If every finitely generated R -algebra A with A/N R -separable has an inertial subalgebra, R is called an *inertial coefficient ring*. The basic properties of inertial subalgebras and inertial coefficient rings can be found in [7].

If I is an ideal of a ring B we call (B, I) an *L. I. pair* (lifting idempotent pair) if every idempotent in the factor ring B/I is the image of an idempotent in B ; if (A, N) is an L. I. pair we say A has the *idempotent lifting property*.

Our main theorem is motivated by a conjecture of E. C. Ingraham that if every finitely generated R -algebra has the idempotent lifting property then R is an inertial coefficient ring. Our proof of the converse of this conjecture has as a corollary that an inertial coefficient ring is a Hensel ring (see [5], [6], and [10] for definition and properties of Hensel rings), recalling the role Hensel local rings have played in generalizations of the Wedderburn Principal Theorem by Azumaya, Ingraham, and W. C. Brown. A second immediate consequence of our main theorem is that when R is an inertial coefficient ring, two inertial subalgebras of an R -algebra A are conjugate under an inner automorphism of A , generalizing Malcev's uniqueness statement to Wedderburn's Principal Theorem.

Received by the editors April 14, 1976.

AMS (MOS) subject classifications (1970). Primary 16A16, 16A32; Secondary 13J15.

Key words and phrases. Inertial coefficient ring, lifting idempotents, Hensel ring, separable algebra.

¹This material appeared in my doctoral dissertation written at Michigan State University. I am deeply indebted to Professor E. C. Ingraham for his guidance in its preparation.

© American Mathematical Society 1977

Our main theorem is proved in three steps. We first consider the case where R is an inertial coefficient ring which contains no idempotents but 0 and 1, (i.e. R is a *connected* ring) and show $R/\text{rad}(R)$ is connected. We next use the Pierce representation of R as the ring of global sections of a sheaf of connected rings [12] to show that if R is any inertial coefficient ring, an idempotent in $R/\text{rad}(R)$ is the image of an idempotent in R . Finally we show that any finitely generated R -algebra has the idempotent lifting property.

THEOREM 1. *Let R be a connected ring. If $R/\text{rad}(R)$ has an idempotent not equal to 0 or 1 then R is not an inertial coefficient ring.*

PROOF. By [9, Proposition 4, p. 54] there exists a $p \in \text{rad}(R)$ such that $x^2 - x + p \in R[x]$ has no root in $\text{rad}(R)$. Let $f(x) = x^3 - x^2 + px$. Reducing the coefficients of $f(x)$ modulo $\text{rad}(R)$ we obtain, $\bar{f}(x) = x^2(x - 1)$, but it is easy to check that there do not exist monic polynomials $g(x)$ and $h(x)$ contained in $R[x]$ such that $f(x) = g(x)h(x)$, with $\bar{g}(x) = x^2$ and $\bar{h}(x) = x - 1$.

Let A be the finitely generated, faithful, free, commutative R -algebra $A = R[x]/\langle f(x) \rangle$. Using the Chinese Remainder Theorem we obtain

$$A / (\text{rad}(R)A) \cong ((R/\text{rad}(R))[x]/\langle x^2 \rangle) \oplus ((R/\text{rad}(R))[x]/\langle x - 1 \rangle).$$

Since $\text{rad}(R) \cdot A \subseteq N$, $A/N \cong R/\text{rad}(R) \oplus R/\text{rad}(R)$, and hence A/N is R -separable.

We shall show that the assumption that A has an inertial subalgebra S leads to a contradiction. By [7, Proposition 2.8, p. 80] if such an S exists, it must be a projective R -module and by [2, proof of Lemma 1, p. 11] $\text{rank}_R S = 2$, where by $\text{rank}_R S$ we mean the dimension of the free R_m -module S_m , for any maximal ideal m of R [3, Theorem 4.12, p. 32].

Case 1. Assume S is a connected ring. By the “projective lifting property” of separable algebras [3, Proposition 2.3, p. 48], A is a projective S -module. Since S is connected, $\text{rank}_S(A)$ is defined, and the multiplicative property of rank [3, Exercise 2, p. 35] yields $3 = \text{rank}_R(A) = \text{rank}_R(S) \text{rank}_S(A) = 2 \text{rank}_S(A)$. Thus 2 divides 3, a contradiction.

Case 2. Assume S has an idempotent $e \neq 0, 1$. Since $\text{rank}_R(Se) = 1$ and since Re is an Re -direct summand of Se [3, Corollary 4.2, p. 56], we have $Se \cong Re$ as rings. Similarly $S(1 - e) \cong R(1 - e)$, so $S \cong Re \oplus R(1 - e)$ (as rings). We now show that we can assume e to be of the form $a_1x + a_2x^2$ where $a_1 \in \text{rad}(R)$ and a_2 is a unit of R such that \bar{a}_2 is an idempotent in $R/\text{rad}(R)$.

First, writing $e = a_0 + a_1x + a_2x^2$ with $a_i \in R$, we have

$$\begin{aligned} 0 = e^2 - e &= (a_0^2 - a_0) + (2a_0a_1 - a_1)x \\ &\quad + (a_1^2 - a_2 + 2a_0a_2)x^2 + (2a_1a_2)x^3 + (a_2^2)x^4. \end{aligned}$$

Now applying the relations $x^3 = x^2 - px$ and $x^4 = (1 - p)x^2 - px$ we ob-

$$0 = (a_0^2 - a_0) + (2a_0a_1 - a_1 - 2a_1a_2p - a_2^2p)x + (a_1^2 - a_2 + 2a_0a_1 + 2a_1a_2 + a_2^2(1 - p))x^2.$$

This equation yields the following relations:

- (1) $a_0^2 = a_0,$
- (2) $2a_0a_1 - a_1 - 2a_1a_2p - a_2^2p = 0,$
- (3) $a_1^2 - a_2 + 2a_0a_2 + 2a_1a_2 + a_2^2(1 - p) = 0.$

Since R is connected, equation (1) implies $a_0 = 0$ or $a_0 = 1$ and equations (2) and (3) imply e or $1 - e$ is of the form $a_1x + a_2x^2$ where $a_1 \in \text{rad}(R)$ and \bar{a}_2 is an idempotent in $R/\text{rad}(R)$.

Second, one can check that $A/N \cong (R/\text{rad } R)[x]/\langle x^2 - x \rangle$. From $A = S + N$ it follows that there exist $r_1, r_2 \in R$ and $n \in N$ such that $x = r_2 + (r_1 - r_2)e + n$. Since $\bar{e} = \bar{a}_2\bar{x}^2$ in $A/(\text{rad}(R)A)$, we have

$$\bar{x} = \bar{r}_2 + (\bar{r}_1 - \bar{r}_2)\bar{a}_2\bar{x}^2 + \bar{n} \quad \text{in } A/(\text{rad}(R)A)$$

and

$$\bar{x} = \bar{r}_2 + (\bar{r}_1 - \bar{r}_2)\bar{a}_2\bar{x} \quad \text{in } A/N,$$

where \bar{a} denotes the image of $a \in A$ in A/N . Hence $\bar{r}_2 = \bar{0}$ and $\bar{1} = \bar{r}_1\bar{a}_2$. Since \bar{a}_2 is both a unit and an idempotent of A/N , $\bar{a}_2 = \bar{1}$. But then by [5, Lemma 1.2, p. 46], $\bar{a}_2 = \bar{1}$ in $A/(\text{rad}(R)A)$ and, hence, $\bar{e} = \bar{x}^2$ and a_2 is a unit of R .

Thus

$$\begin{aligned} A &= Ae \oplus A(1 - e) = \langle a_2x^2 + a_1x \rangle \oplus \langle a_2x^2 + a_1x - 1 \rangle \\ &= \langle x^2 + a_2^{-1}a_1x \rangle \oplus \langle x^2 + a_2^{-1}a_1x - a_2^{-1} \rangle. \end{aligned}$$

Furthermore

$$x^2 + a_2^{-1}a_1x \equiv x^2 \quad \text{modulo } \text{rad}(R)A$$

and

$$x^2 + a_2^{-1}a_1x - a_2^{-1} \equiv x^2 - 1 \quad \text{modulo } \text{rad}(R)A.$$

Finally $\langle x^2 - 1 \rangle = \langle x - 1 \rangle$ in $A/\text{rad}(R)A$, since $\langle x^2 - 1 \rangle \subseteq \langle x - 1 \rangle$ and $(1 - x)(x^2 - 1) = (x - 1)$ in $A/(\text{rad}(R)A)$ implies $\langle x - 1 \rangle \subseteq \langle x^2 - 1 \rangle$. Thus by [5, proof of Theorem 3.1, p. 54] there exist monic polynomials $g(x)$ and $h(x)$ in $R[x]$ such that $x^3 - x^2 + px = g(x) \cdot h(x)$ with $\bar{g}(x) = x^2$ and $\bar{h}(x) = x - 1$ in $(R/\text{rad } R)[x]$. This contradicts the choice of p and completes the proof.

Theorem 1 states that if R is a connected inertial coefficient ring then $R/\text{rad}(R)$ is connected, and so R trivially has the idempotent lifting property. We next use the Pierce representation of R to extend Theorem 1 to an arbitrary inertial coefficient ring.

The decomposition space $\chi(R)$ of R can be viewed as the quotient space

obtained from the set of prime ideals of R endowed with the Zariski topology by identifying the points in each connected component. It can be shown [11, Proposition II.12, p. 30] that the open, closed subsets of $X(R)$ form a basis for the topology on $X(R)$, and that any open, closed subset of $X(R)$ is of the form $N(e) = \{x \in X(R) : x \subseteq V(R(1 - e))\}$, where e is an idempotent of R and $V(R(1 - e))$ is the set of prime ideals of R which contain the ideal $R(1 - e)$.

Pierce represents R as the ring of global sections of a sheaf over $X(R)$ where at each point $x \in X(R)$ the stalk $R_x = R/I(x)$ is a homomorphic image of R and is a connected ring. If M is an R -module, $M_x = M \otimes_R R_x \cong M/I(x)M$ is an R_x -module; if $m \in M$, m_x denotes the image of m in M_x .

PROPOSITION 2. *If (A_x, N_x) is an L. I. pair for all $x \in X(R)$, then (A, N) is an L. I. pair.*

PROOF. Let $u \in A$ be such that $u^2 - u \in N$. We shall produce an idempotent $e \in N$ such that $u - e \in N$.

Since R_x is a flat R -module, without ambiguity we can let N_x denote the image of N under the canonical homomorphism $A \rightarrow A/I(x)A = A_x$. Now $u_x = u + I(x)A$ is an element of A_x such that $u_x + N_x$ is an idempotent element of A_x/N_x . Since (A_x, N_x) is an L. I. pair and since an idempotent in A_x can be lifted to an idempotent in A [11, Proposition II.20, p. 34], there exists an idempotent $f(x) \in A$ such that $u_x = [f(x)]_x + [n(x)]_x$ for some $n(x) \in N$. By [11, Proposition II.16, p. 32] for each $x \in X(R)$ there exists an idempotent $e(x) \in R$ such that $ue(x) = f(x)e(x) + n(x)e(x)$ and $u_y = [f(x)]_y + [h(x)]_y$ for all $y \in N(e(x))$.

By the “partition property” of $X(R)$, the open cover $\{N(e(x))\}$ of $X(R)$ has a finite refinement of disjoint, open, closed sets $\{N(e_i)\}_{i=1}^m$, for some idempotents $e_i \in R$, which covers $X(R)$. Since $\{N(e_i)\}_{i=1}^m$ is a cover of $X(R)$ by disjoint sets, it follows that $1 = \sum_{i=1}^m e_i$ and $e_i e_j = 0$ for each $i \neq j$. Furthermore $N(e_i) \subseteq N(e(x_i))$ for some $x_i \in X(R)$, and thus it can be shown that $e_i e(x_i) = e_i$; hence, $ue_i = f_i e_i + n_i e_i$ where $f_i = f(x_i)$ and $n_i = n(x_i)$. Since the e_i are pairwise orthogonal idempotents and each f_i is an idempotent, then $e = \sum_{i=1}^m f_i e_i$ is an idempotent. Now

$$u = \sum_{i=1}^m ue_i = \sum_{i=1}^m (f_i e_i + n_i e_i) = e + \sum_{i=1}^m n_i e_i,$$

and therefore $u - e \in N$.

COROLLARY 3. *If R is an inertial coefficient ring then $(R, \text{rad}(R))$ is an L. I. pair.*

PROOF. Each stalk R_x is a connected ring and is an inertial coefficient ring because it is a homomorphic image of R . By Theorem 1, $(R_x, \text{rad}(R_x))$ is an L. I. pair for every $x \in X(R)$. Since $(\text{rad } R)_x \subseteq \text{rad}(R_x)$ and R_x is a commutative ring, then $(R_x, (\text{rad } R)_x)$ is an L. I. pair for every $x \in X(R)$ [5,

Corollary 1.3, p. 46]; hence by Proposition 2, $(R, \text{rad}(R))$ is an L. I. pair.

We are now able to prove the general case:

THEOREM 4. *Let R be an inertial coefficient ring and A a finitely generated R -algebra. Then (A, N) is an L. I. pair.*

PROOF. By replacing $R/\text{annih}_R A$ by R we may assume that A is a faithful R -algebra.

Let $u \in A$ be such that $u^2 - u \in N$. We must show there exists an idempotent $e \in A$ such that $u - e \in N$. Let B denote the R -subalgebra of A generated by u . Now $B/B \cap N$, being a homomorphic image of $(R/\text{rad } R)[x]/\langle x^2 - x \rangle$, is a separable R -algebra. Since B is a finitely generated R -algebra, $B \cap N \subseteq \text{rad}(B)$ [1, Corollary, p. 126], and hence $B/\text{rad}(B)$ is a separable R -algebra. Since B is a commutative R -algebra, by [7, see proof of Proposition 3.3, p. 85] B is an inertial coefficient ring, and thus by Corollary 3 $(B, \text{rad}(B))$ is an L. I. pair. It follows from the commutativity of B that $(B, N \cap B)$ is an L. I. pair. Then $u^2 - u \in N \cap B$ implies that there exists an idempotent $e \in B$ such that $u - e \in N \cap B$. Then $e \in A$ and $u - e \in N$.

COROLLARY 5. *An inertial coefficient ring is a Hensel ring.*

PROOF. To show R is a Hensel ring it suffices to show that for any finitely generated, commutative, free R -algebra A , $(A, \text{rad}(R)A)$ is an L. I. pair [5, Theorem 4.1, p. 55]. Since R is an inertial coefficient ring, (A, N) is an L. I. pair, and since $\text{rad}(R)A \subseteq N$ and A is commutative, then $(A, \text{rad}(R)A)$ is an L. I. pair.

The following corollary to Theorem 4 follows immediately from [8, Corollary 2, p. 556].

COROLLARY 6. *If R is an inertial coefficient ring and if S and T are two inertial subalgebras of a finitely generated R -algebra A , then $T = (1 + n)^{-1} \cdot S(1 + n)$ for some $n \in N$.*

N. S. Ford has given an example of a ring R and an R -algebra A such that S and T are not isomorphic [4].

The following corollary to Theorem 4 follows immediately from [8, Theorem 2, p. 554].

COROLLARY 7. *The following are equivalent properties of a commutative ring R :*

(i) *For all finitely generated, commutative R -algebras A such that A/N is R -separable, there exists a separable R -subalgebra S of A such that $A = S + N$.*

(ii) *All finitely generated, commutative R -algebras have the idempotent lifting property.*

algebras A in Corollary 7 to commutative R -algebras, for then the lifting of idempotents from A/N to A in all finitely generated R -algebras A is equivalent to the lifting of the separability of A/N to a separable subalgebra S of A in all finitely generated R -algebras.

REFERENCES

1. G. Azumaya, *On maximally central algebras*, Nagoya Math. J. **2** (1951), 119–150. MR **12**, 669.
2. W. C. Brown and E. C. Ingraham, *A characterization of semilocal inertial coefficient rings*, Proc. Amer. Math. Soc. **26** (1970), 10–14. MR **41** #5354.
3. F. DeMeyer and E. C. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Math., vol. 181, Springer-Verlag, Berlin and New York, 1971. MR **43** #6199.
4. N. S. Ford, Ph. D. Dissertation, Michigan State Univ., July 1972.
5. S. Greco, *Algebras over nonlocal Hensel rings*, J. Algebra **8** (1968), 45–59. MR **36** #1435.
6. ———, *Algebras over nonlocal Hensel rings. II*, J. Algebra **13** (1969), 48–56. MR **39** #5629.
7. E. C. Ingraham, *Inertial subalgebras of algebras over commutative rings*, Trans. Amer. Math. Soc. **124** (1966), 77–93. MR **34** #209.
8. ———, *On the existence and conjugacy of inertial subalgebras*, J. Algebra **31** (1974), 547–556. MR **50** #2245.
9. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1956. MR **18**, 373.
10. J.-P. Lafon, *Anneaux Henséliens*, Bull. Soc. Math. France **91** (1963), 77–107. MR **27** #174.
11. A. R. Magid, *The separable Galois theory of commutative rings*, Dekker, New York, 1974. MR **50** #4563.
12. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. No. 70, (1967). MR **36** #151.

DEPARTMENT OF MATHEMATICS, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NORTH CAROLINA 27109