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# INERTIAL COEFFICIENT RINGS AND THE IDEMPOTENT LIFTING PROPERTY[1]

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ABSTRACT. A commutative ring $R$ with identity is called an *inertial coefficient ring* if every finitely generated $R$-algebra $A$ with $A/N$ separable over $R$ contains a separable $R$-subalgebra $S$ of $A$ such that $A = S + N$, where $N$ is the Jacobson radical of $A$. We say $A$ has the *idempotent lifting property* if every idempotent in $A/N$ is the image of an idempotent in $A$. Our main theorem is that any finitely generated algebra over an inertial coefficient ring has the idempotent lifting property.

All rings contain an identity; all subrings contain the identity of the overring; all homomorphisms preserve the identity. Throughout $R$ denotes a commutative ring and $A$ an $R$-algebra which is finitely generated as an $R$-module. The Jacobson radical of a ring $B$ is denoted $\operatorname{rad}(B)$ and throughout $\operatorname{rad}(A) = N$. A separable $R$-subalgebra $S$ of $A$ such that $A = S + N$ is called an *inertial subalgebra.* If every finitely generated $R$-algebra $A$ with $A/N$ $R$-separable has an inertial subalgebra, $R$ is called an *inertial coefficient ring*. The basic properties of inertial subalgebras and inertial coefficient rings can be found in [7].

If $I$ is an ideal of a ring $B$ we call $(B, I)$ an *L. I. pair* (lifting idempotent pair) if every idempotent in the factor ring $B/I$ is the image of an idempotent in $B$; if $(A, N)$ is an L. I. pair we say $A$ has the *idempotent lifting property*.

Our main theorem is motivated by a conjecture of E. C. Ingraham that if every finitely generated $R$-algebra has the idempotent lifting property then $R$ is an inertial coefficient ring. Our proof of the converse of this conjecture has as a corollary that an inertial coefficient ring is a Hensel ring (see [5], [6], and [10] for definition and properties of Hensel rings), recalling the role Hensel local rings have played in generalizations of the Wedderburn Principal Theorem by Azumaya, Ingraham, and W. C. Brown. A second immediate consequence of our main theorem is that when $R$ is an inertial coefficient ring, two inertial subalgebras of an $R$-algebra $A$ are conjugate under an inner automorphism of $A$, generalizing Malcev's uniqueness statement to Wedderburn's Principal Theorem.

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Our main theorem is proved in three steps. We first consider the case where $R$ is an inertial coefficient ring which contains no idempotents but 0 and 1, (i.e. $R$ is a *connected* ring) and show $R/\mathrm{rad}(R)$ is connected. We next use the Pierce representation of $R$ as the ring of global sections of a sheaf of connected rings [12] to show that if $R$ is any inertial coefficient ring, an idempotent in $R/\mathrm{rad}(R)$ is the image of an idempotent in $R$. Finally we show that any finitely generated $R$-algebra has the idempotent lifting property.

THEOREM 1. *Let $R$ be a connected ring. If $R/\mathrm{rad}(R)$ has an idempotent not equal to* 0 *or* 1 *then $R$ is not an inertial coefficient ring.*

PROOF. By [9, Proposition 4, p. 54] there exists a $p \in \mathrm{rad}(R)$ such that $x^2 - x + p \in R[x]$ has no root in $\mathrm{rad}(R)$. Let $f(x) = x^3 - x^2 + px$. Reducing the coefficients of $f(x)$ modulo $\mathrm{rad}(R)$ we obtain, $\bar{f}(x) = x^2(x - 1)$, but it is easy to check that there do not exist monic polynomials $g(x)$ and $h(x)$ contained in $R[x]$ such that $f(x) = g(x)h(x)$, with $\bar{g}(x) = x^2$ and $\bar{h}(x) = x - 1$.

Let $A$ be the finitely generated, faithful, free, commutative $R$-algebra $A = R[x]/\langle f(x)\rangle$. Using the Chinese Remainder Theorem we obtain

$$A/(\mathrm{rad}(R)A) \cong ((R/\mathrm{rad}R)[x]/\langle x^2\rangle) \oplus ((R/\mathrm{rad}R)[x]/\langle x - 1\rangle).$$

Since $\mathrm{rad}(R) \cdot A \subseteq N$, $A/N \cong R/\mathrm{rad}(R) \oplus R/\mathrm{rad}(R)$, and hence $A/N$ is $R$-separable.

We shall show that the assumption that $A$ has an inertial subalgebra $S$ leads to a contradiction. By [7, Proposition 2.8, p. 80] if such an $S$ exists, it must be a projective $R$-module and by [2, proof of Lemma 1, p. 11] $\mathrm{rank}_R\, S = 2$, where by $\mathrm{rank}_R\, S$ we mean the dimension of the free $R_m$-module $S_m$, for any maximal ideal $m$ of $R$ [3, Theorem 4.12, p. 32].

*Case* 1. Assume $S$ is a connected ring. By the "projective lifting property" of separable algebras [3, Proposition 2.3, p. 48], $A$ is a projective $S$-module. Since $S$ is connected, $\mathrm{rank}_S(A)$ is defined, and the multiplicative property of rank [3, Exercise 2, p. 35] yields $3 = \mathrm{rank}_R(A) = \mathrm{rank}_R(S)\ \mathrm{rank}_S(A) = 2\ \mathrm{rank}_S(A)$. Thus 2 divides 3, a contradiction.

*Case* 2. Assume $S$ has an idempotent $e \neq 0, 1$. Since $\mathrm{rank}_R(Se) = 1$ and since $Re$ is an $Re$-direct summand of $Se$ [3, Corollary 4.2, p. 56], we have $Se \cong Re$ as rings. Similarly $S(1 - e) \cong R(1 - e)$, so $S \cong Re \oplus R(1 - e)$ (as rings). We now show that we can assume $e$ to be of the form $a_1x + a_2x^2$ where $a_1 \in \mathrm{rad}(R)$ and $a_2$ is a unit of $R$ such that $\bar{a}_2$ is an idempotent in $R/\mathrm{rad}(R)$.

First, writing $e = a_0 + a_1x + a_2x^2$ with $a_i \in R$, we have

$$\begin{aligned} 0 = e^2 - e &= (a_0^2 - a_0) + (2a_0a_1 - a_1)x \\ &\quad + (a_1^2 - a_2 + 2a_0a_2)x^2 + (2a_1a_2)x^3 + (a_2^2)x^4. \end{aligned}$$

Now applying the relations $x^3 = x^2 - px$ and $x^4 = (1 - p)x^2 - px$ we obtain

$$0 = \left(a_0^2 - a_0\right) + \left(2a_0a_1 - a_1 - 2a_1a_2p - a_2^2p\right)x$$
$$+ \left(a_1^2 - a_2 + 2a_0a_1 + 2a_1a_2 + a_2^2(1-p)\right)x^2.$$

This equation yields the following relations:

(1) $$a_0^2 = a_0,$$

(2) $$2a_0a_1 - a_1 - 2a_1a_2p - a_0^2p = 0,$$

(3) $$a_1^2 - a_2 + 2a_0a_2 + 2a_1a_2 + a_2^2(1-p) = 0.$$

Since $R$ is connected, equation (1) implies $a_0 = 0$ or $a_0 = 1$ and equations (2) and (3) imply $e$ or $1 - e$ is of the form $a_1x + a_2x^2$ where $a_1 \in \operatorname{rad}(R)$ and $\bar{a}_2$ is an idempotent in $R/\operatorname{rad}(R)$.

Second, one can check that $A/N \cong (R/\operatorname{rad} R)[x]/\langle x^2 - x\rangle$. From $A = S + N$ it follows that there exist $r_1, r_2 \in R$ and $n \in N$ such that $x = r_2 + (r_1 - r_2)e + n$. Since $\bar{e} = \bar{a}_2\bar{x}^2$ in $A/(\operatorname{rad}(R)A)$, we have

$$\bar{x} = \bar{r}_2 + (\bar{r}_1 - \bar{r}_2)\bar{a}_2\bar{x}^2 + \bar{n} \quad \text{in } A/\,(\operatorname{rad}(R)A)$$

and

$$\bar{\bar{x}} = \bar{\bar{r}}_2 + \left(\bar{\bar{r}}_1 - \bar{\bar{r}}_2\right)\bar{\bar{a}}_2\bar{\bar{x}} \quad \text{in } A/N,$$

where $\bar{\bar{a}}$ denotes the image of $a \in A$ in $A/N$. Hence $\bar{\bar{r}}_2 = \bar{\bar{0}}$ and $\bar{\bar{1}} = \bar{\bar{r}}_1\bar{\bar{a}}_2$. Since $\bar{\bar{a}}_2$ is both a unit and an idempotent of $A/N$, $\bar{\bar{a}}_2 = \bar{\bar{1}}$. But then by [5, Lemma 1.2, p. 46], $\bar{a}_2 = \bar{1}$ in $A/(\operatorname{rad}(R)A)$ and, hence, $\bar{e} = \bar{x}^2$ and $a_2$ is a unit of $R$.

Thus

$$A = Ae \oplus A(1-e) = \langle a_2x^2 + a_1x\rangle \oplus \langle a_2x^2 + a_1x - 1\rangle$$
$$= \langle x^2 + a_2^{-1}a_1x\rangle \oplus \langle x^2 + a_2^{-1}a_1x - a_2^{-1}\rangle.$$

Furthermore

$$x^2 + a_2^{-1}a_1x \equiv x^2 \quad \text{modulo } \operatorname{rad}(R)A$$

and

$$x^2 + a_2^{-1}a_1x - a_2^{-1} \equiv x^2 - 1 \quad \text{modulo } \operatorname{rad}(R)A.$$

Finally $\langle x^2 - 1\rangle = \langle x - 1\rangle$ in $A/\operatorname{rad}(R)A$, since $\langle x^2 - 1\rangle \subseteq \langle x - 1\rangle$ and $(1 - x)(x^2 - 1) = (x - 1)$ in $A/(\operatorname{rad}(R)A)$ implies $\langle x - 1\rangle \subseteq \langle x^2 - 1\rangle$. Thus by [5, proof of Theorem 3.1, p. 54] there exist monic polynomials $g(x)$ and $h(x)$ in $R[x]$ such that $x^3 - x^2 + px = g(x) \cdot h(x)$ with $\bar{g}(x) = x^2$ and $\bar{h}(x) = x - 1$ in $(R/\operatorname{rad} R)[x]$. This contradicts the choice of $p$ and completes the proof.

Theorem 1 states that if $R$ is a connected inertial coefficient ring then $R/\operatorname{rad}(R)$ is connected, and so $R$ trivially has the idempotent lifting property. We next use the Pierce representation of $R$ to extend Theorem 1 to an arbitrary inertial coefficient ring.

The decomposition space $X(R)$ of $R$ can be viewed as the quotient space obtained from the set of prime ideals of $R$ endowed with the Zariski topology by identifying the points in each connected component. It can be shown [11, Proposition II.12, p. 30] that the open, closed subsets of $X(R)$ form a basis for the topology on $X(R)$, and that any open, closed subset of $X(R)$ is of the form $N(e) = \{x \in X(R)\colon x \subseteq V(R(1-e))\}$, where $e$ is an idempotent of $R$ and $V(R(1-e))$ is the set of prime ideals of $R$ which contain the ideal $R(1-e)$.

Pierce represents $R$ as the ring of global sections of a sheaf over $X(R)$ where at each point $x \in X(R)$ the stalk $R_x = R/I(x)$ is a homomorphic image of $R$ and is a connected ring. If $M$ is an $R$-module, $M_x = M \otimes_R R_x \cong M/I(x)M$ is an $R_x$-module; if $m \in M$, $m_x$ denotes the image of $m$ in $M_x$.

PROPOSITION 2. *If $(A_x, N_x)$ is an L. I. pair for all $x \in X(R)$, then $(A, N)$ is an L. I. pair.*

PROOF. Let $u \in A$ be such that $u^2 - u \in N$. We shall produce an idempotent $e \in N$ such that $u - e \in N$.

Since $R_x$ is a flat $R$-module, without ambiguity we can let $N_x$ denote the image of $N$ under the canonical homomorphism $A \to A/I(x)A = A_x$. Now $u_x = u + I(x)A$ is an element of $A_x$ such that $u_x + N_x$ is an idempotent element of $A_x/N_x$. Since $(A_x, N_x)$ is an L. I. pair and since an idempotent in $A_x$ can be lifted to an idempotent in $A$ [11, Proposition II.20, p. 34], there exists an idempotent $f(x) \in A$ such that $u_x = [f(x)]_x + [n(x)]_x$ for some $n(x) \in N$. By [11, Proposition II.16, p. 32] for each $x \in X(R)$ there exists an idempotent $e(x) \in R$ such that $ue(x) = f(x)e(x) + n(x)e(x)$ and $u_y = [f(x)]_y + [h(x)]_y$ for all $y \in N(e(x))$.

By the "partition property" of $X(R)$, the open cover $\{N(e(x))\}$ of $X(R)$ has a finite refinement of disjoint, open, closed sets $\{N(e_i)\}_{i=1}^m$, for some idempotents $e_i \in R$, which covers $X(R)$. Since $\{N(e_i)\}_{i=1}^m$ is a cover of $X(R)$ by disjoint sets, it follows that $1 = \sum_{i=1}^m e_i$ and $e_ie_j = 0$ for each $i \neq j$. Furthermore $N(e_i) \subseteq N(e(x_i))$ for some $x_i \in X(R)$, and thus it can be shown that $e_ie(x_i) = e_i$; hence, $ue_i = f_ie_i + n_ie_i$ where $f_i = f(x_i)$ and $n_i = n(x_i)$. Since the $e_i$ are pairwise orthogonal idempotents and each $f_i$ is an idempotent, then $e = \sum_{i=1}^m f_ie_i$ is an idempotent. Now

$$u = \sum_{i=1}^m ue_i = \sum_{i=1}^m (f_ie_i + n_ie_i) = e + \sum_{i=1}^m n_ie_i,$$

and therefore $u - e \in N$.

COROLLARY 3. *If $R$ is an inertial coefficient ring then $(R, \operatorname{rad}(R))$ is an L. I. pair.*

PROOF. Each stalk $R_x$ is a connected ring and is an inertial coefficient ring because it is a homomorphic image of $R$. By Theorem 1, $(R_x, \operatorname{rad}(R_x))$ is an L. I. pair for every $x \in X(R)$. Since $(\operatorname{rad} R)_x \subseteq \operatorname{rad}(R_x)$ and $R_x$ is a commutative ring, then $(R_x, (\operatorname{rad} R)_x)$ is an L. I. pair for every $x \in X(R)$ [5, Corollary 1.3, p. 46]; hence by Proposition 2, $(R, \operatorname{rad}(R))$ is an L. I. pair.

We are now able to prove the general case:

THEOREM 4. *Let $R$ be an inertial coefficient ring and $A$ a finitely generated $R$-algebra. Then $(A, N)$ is an L. I. pair.*

PROOF. By replacing $R/\operatorname{annih}_R A$ by $R$ we may assume that $A$ is a faithful $R$-algebra.

Let $u \in A$ be such that $u^2 - u \in N$. We must show there exists an idempotent $e \in A$ such that $u - e \in N$. Let $B$ denote the $R$-subalgebra of $A$ generated by $u$. Now $B/B \cap N$, being a homomorphic image of $(R/\operatorname{rad} R)[x]/\langle x^2 - x\rangle$, is a separable $R$-algebra. Since $B$ is a finitely generated $R$-algebra, $B \cap N \subseteq \operatorname{rad}(B)$ [1, Corollary, p. 126], and hence $B/\operatorname{rad}(B)$ is a separable $R$-algebra. Since $B$ is a commutative $R$-algebra, by [7, see proof of Proposition 3.3, p. 85] $B$ is an inertial coefficient ring, and thus by Corollary 3 $(B, \operatorname{rad}(B))$ is an L. I. pair. It follows from the commutativity of $B$ that $(B, N \cap B)$ is an L. I. pair. Then $u^2 - u \in N \cap B$ implies that there exists an idempotent $e \in B$ such that $u - e \in N \cap B$. Then $e \in A$ and $u - e \in N$.

COROLLARY 5. *An inertial coefficient ring is a Hensel ring.*

PROOF. To show $R$ is a Hensel ring it suffices to show that for any finitely generated, commutative, free $R$-algebra $A$, $(A, \operatorname{rad}(R)A)$ is an L. I. pair [5, Theorem 4.1, p. 55]. Since $R$ is an inertial coefficient ring, $(A, N)$ is an L. I. pair, and since $\operatorname{rad}(R)A \subseteq N$ and $A$ is commutative, then $(A, \operatorname{rad}(R)A)$ is an L. I. pair.

The following corollary to Theorem 4 follows immediately from [8, Corollary 2, p. 556].

COROLLARY 6. *If $R$ is an inertial coefficient ring and if $S$ and $T$ are two inertial subalgebras of a finitely generated $R$-algebra $A$, then $T = (1 + n)^{-1} \cdot S(1 + n)$ for some $n \in N$.*

N. S. Ford has given an example of a ring $R$ and an $R$-algebra $A$ such that $S$ and $T$ are not isomorphic [4].

The following corollary to Theorem 4 follows immediately from [8, Theorem 2, p. 554].

COROLLARY 7. *The following are equivalent properties of a commutative ring $R$:*

(i) *For all finitely generated, commutative $R$-algebras $A$ such that $A/N$ is $R$-separable, there exists a separable $R$-subalgebra $S$ of $A$ such that $A = S + N$.*

(ii) *All finitely generated, commutative $R$-algebras have the idempotent lifting property.*

If Ingraham's conjecture is true then it is unnecessary to restrict the algebras $A$ in Corollary 7 to commutative $R$-algebras, for then the lifting of idempotents from $A/N$ to $A$ in all finitely generated $R$-algebras $A$ is equivalent to the lifting of the separability of $A/N$ to a separable subalgebra $S$ of $A$ in all finitely generated $R$-algebras.

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