INERTIAL COEFFICIENT RINGS AND THE IDEMPOTENT LIFTING PROPERTY

ELLEN E. KIRKMAN

Abstract. A commutative ring R with identity is called an inertial coefficient ring if every finitely generated R-algebra A with A/N separable over R contains a separable R-subalgebra S of A such that A = S + N, where N is the Jacobson radical of A. We say A has the idempotent lifting property if every idempotent in A/N is the image of an idempotent in A. Our main theorem is that any finitely generated algebra over an inertial coefficient ring has the idempotent lifting property.

All rings contain an identity; all subrings contain the identity of the overring; all homomorphisms preserve the identity. Throughout R denotes a commutative ring and A an R-algebra which is finitely generated as an R-module. The Jacobson radical of a ring B is denoted rad(B) and throughout rad(A) = N. A separable R-subalgebra S of A such that A = S + N is called an inertial subalgebra. If every finitely generated R-algebra A with A/N R-separable has an inertial subalgebra, R is called an inertial coefficient ring. The basic properties of inertial subalgebras and inertial coefficient rings can be found in [7].

If I is an ideal of a ring B we call (B, I) an L.I. pair (lifting idempotent pair) if every idempotent in the factor ring B/I is the image of an idempotent in B; if (A, N) is an L.I. pair we say A has the idempotent lifting property.

Our main theorem is motivated by a conjecture of E. C. Ingraham that if every finitely generated R-algebra has the idempotent lifting property then R is an inertial coefficient ring. Our proof of the converse of this conjecture has as a corollary that an inertial coefficient ring is a Hensel ring (see [5], [6], and [10] for definition and properties of Hensel rings), recalling the role Hensel local rings have played in generalizations of the Wedderburn Principal Theorem by Azumaya, Ingraham, and W. C. Brown. A second immediate consequence of our main theorem is that when R is an inertial coefficient ring, two inertial subalgebras of an R-algebra A are conjugate under an inner automorphism of A, generalizing Malcev's uniqueness statement to Wedderburn's Principal Theorem.

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1This material appeared in my doctoral dissertation written at Michigan State University. I am deeply indebted to Professor E. C. Ingraham for his guidance in its preparation.
Our main theorem is proved in three steps. We first consider the case where $R$ is an inertial coefficient ring which contains no idempotents but 0 and 1, (i.e. $R$ is a connected ring) and show $R/\text{rad}(R)$ is connected. We next use the Pierce representation of $R$ as the ring of global sections of a sheaf of connected rings [12] to show that if $R$ is any inertial coefficient ring, an idempotent in $R/\text{rad}(R)$ is the image of an idempotent in $R$. Finally we show that any finitely generated $R$-algebra has the idempotent lifting property.

**Theorem 1.** Let $R$ be a connected ring. If $R/\text{rad}(R)$ has an idempotent not equal to 0 or 1 then $R$ is not an inertial coefficient ring.

**Proof.** By [9, Proposition 4, p. 54] there exists a $p \in \text{rad}(R)$ such that $x^2 - x + p \in R[x]$ has no root in $\text{rad}(R)$. Let $f(x) = x^3 - x^2 + px$. Reducing the coefficients of $f(x)$ modulo $\text{rad}(R)$ we obtain $\bar{f}(x) = x^2(x - 1)$, but it is easy to check that there do not exist monic polynomials $g(x)$ and $h(x)$ contained in $R[x]$ such that $f(x) = g(x)h(x)$, with $\bar{g}(x) = x^2$ and $\bar{h}(x) = x - 1$.

Let $A$ be the finitely generated, faithful, free, commutative $R$-algebra $A = R[x]/\langle f(x) \rangle$. Using the Chinese Remainder Theorem we obtain

$$A/\langle \text{rad}(R)A \rangle \cong ((R/\text{rad}(R))[x]/\langle x^2 \rangle) \oplus ((R/\text{rad}(R))[x]/\langle x - 1 \rangle).$$

Since $\text{rad}(R) \cdot A \subseteq N$, $A/N \cong R/\text{rad}(R) \oplus R/\text{rad}(R)$, and hence $A/N$ is $R$-separable.

We shall show that the assumption that $A$ has an inertial subalgebra $S$ leads to a contradiction. By [7, Proposition 2.8, p. 80] if such an $S$ exists, it must be a projective $R$-module and by [2, proof of Lemma 1, p. 11] $\text{rank}_R S = 2$, where by $\text{rank}_R S$ we mean the dimension of the free $R_m$-module $S_m$, for any maximal ideal $m$ of $R$ [3, Theorem 4.12, p. 32].

**Case 1.** Assume $S$ is a connected ring. By the “projective lifting property” of separable algebras [3, Proposition 2.3, p. 48], $A$ is a projective $S$-module. Since $S$ is connected, $\text{rank}_S(A)$ is defined, and the multiplicative property of rank [3, Exercise 2, p. 35] yields $3 = \text{rank}_R(A) = \text{rank}_R(S) \cdot \text{rank}_S(A) = 2 \cdot \text{rank}_S(A)$. Thus 2 divides 3, a contradiction.

**Case 2.** Assume $S$ has an idempotent $e \neq 0, 1$. Since $\text{rank}_R(Se) = 1$ and since $Re$ is an $R$-direct summand of $Se$ [3, Corollary 4.2, p. 56], we have $Se \cong Re$ as rings. Similarly $S(1 - e) \cong R(1 - e)$, so $S \cong Re \oplus R(1 - e)$ (as rings). We now show that we can assume $e$ to be of the form $a_1 x + a_2 x^2$ where $a_1 \in \text{rad}(R)$ and $a_2$ is a unit of $R$ such that $\bar{a}_2$ is an idempotent in $R/\text{rad}(R)$.

First, writing $e = a_0 + a_1 x + a_2 x^2$ with $a_i \in R$, we have

$$0 = e^2 - e = (a_0^2 - a_0) + (2a_0 a_1 - a_1)x + (a_1^2 - a_2 + 2a_0 a_2)x^2 + (2a_1 a_2)x^3 + (a_2^2)x^4.$$  

Now applying the relations $x^3 = x^2 - px$ and $x^4 = (1 - p)x^2 - px$ we obtain
\[ 0 = (a_0^2 - a_0) + (2a_0a_1 - a_1 - 2a_1a_2p - a_2^2p)x \\
+ (a_1^2 - a_2 + 2a_0a_1 + 2a_1a_2 + a_2^2(1 - p))x^2. \]

This equation yields the following relations:

(1) \[ a_0^2 = a_0, \]

(2) \[ 2a_0a_1 - a_1 - 2a_1a_2p - a_2^2p = 0, \]

(3) \[ a_1^2 - a_2 + 2a_0a_2 + 2a_1a_2 + a_2^2(1 - p) = 0. \]

Since \( R \) is connected, equation (1) implies \( a_0 = 0 \) or \( a_0 = 1 \) and equations (2) and (3) imply \( e \) or \( 1 - e \) is of the form \( a_1x + a_2x^2 \) where \( a_1 \in \text{rad}(R) \) and \( a_2 \) is an idempotent in \( R/\text{rad}(R) \).

Second, one can check that \( A/N = (R/\text{rad } R)[x]/\langle x^2 - x \rangle \). From \( A = S + N \) it follows that there exist \( r_1, r_2 \in R \) and \( n \in N \) such that \( x = r_2 + (r_1 - r_2)e + n \). Since \( e = a_2^2x^2 \) in \( A/(\text{rad}(R)A) \), we have

\[ x = r_2 + (r_1 - r_2)a_2^2x^2 + n \quad \text{in } A/(\text{rad}(R)A) \]

and

\[ \bar{x} = \bar{r}_2 + (\bar{r}_1 - \bar{r}_2)a_2^2 \bar{x}^2 \quad \text{in } A/N, \]

where \( \bar{a} \) denotes the image of \( a \in A \) in \( A/N \). Hence \( \bar{r}_2 = 0 \) and \( 1 = \bar{r}_1\bar{a}_2 \).

Since \( a_2 \) is both a unit and an idempotent of \( A/N \), \( \bar{a}_2 = 1 \) in \( A/(\text{rad}(R)A) \) and, hence, \( e = x^2 \) and \( a_2 \) is a unit of \( R \).

Thus

\[ A = Ae \oplus A(1 - e) = \langle a_2x^2 + a_1x \rangle \oplus \langle a_2x^2 + a_1x - 1 \rangle \]

\[ = \langle x^2 + a_2^{-1}a_1x \rangle \oplus \langle x^2 + a_2^{-1}a_1x - a_2^{-1} \rangle. \]

Furthermore

\[ x^2 + a_2^{-1}a_1x \equiv x^2 \pmod{\text{rad}(R)A} \]

and

\[ x^2 + a_2^{-1}a_1x - a_2^{-1} \equiv x^2 - 1 \pmod{\text{rad}(R)A}. \]

Finally \( \langle x^2 - 1 \rangle = \langle x - 1 \rangle \) in \( A/(\text{rad}(R)A) \), since \( \langle x^2 - 1 \rangle \subseteq \langle x - 1 \rangle \) and \( (1 - x)(x^2 - 1) = (x - 1) \) in \( A/(\text{rad}(R)A) \) implies \( \langle x - 1 \rangle \subseteq \langle x^2 - 1 \rangle \). Thus by [5, proof of Theorem 3.1, p. 54] there exist monic polynomials \( g(x) \) and \( h(x) \) in \( R[x] \) such that \( x^3 - x^2 + px = g(x) \cdot h(x) \) with \( \bar{g}(x) = x^2 \) and \( \bar{h}(x) = x - 1 \) in \( (R/\text{rad } R)[x] \). This contradicts the choice of \( p \) and completes the proof.

Theorem 1 states that if \( R \) is a connected inertial coefficient ring then \( R/\text{rad}(R) \) is connected, and so \( R \) trivially has the idempotent lifting property. We next use the Pierce representation of \( R \) to extend Theorem 1 to an arbitrary inertial coefficient ring.

The decomposition space \( X(R) \) of \( R \) can be viewed as the quotient space
obtained from the set of prime ideals of $R$ endowed with the Zariski topology by identifying the points in each connected component. It can be shown [11, Proposition II.12, p. 30] that the open, closed subsets of $X(R)$ form a basis for the topology on $X(R)$, and that any open, closed subset of $X(R)$ is of the form $N(e) = \{x \in X(R) : x \subseteq V(R(1 - e))\}$, where $e$ is an idempotent of $R$ and $V(R(1 - e))$ is the set of prime ideals of $R$ which contain the ideal $R(1 - e)$.

Pierce represents $R$ as the ring of global sections of a sheaf over $X(R)$ where at each point $x \in X(R)$ the stalk $R_x = R/I(x)$ is a homomorphic image of $R$ and is a connected ring. If $M$ is an $R$-module, $M_x = M \otimes_R R_x \cong M/I(x)M$ is an $R_x$-module; if $m \in M$, $m_x$ denotes the image of $m$ in $M_x$.

**Proposition 2.** If $(A_x, N_x)$ is an L. I. pair for all $x \in X(R)$, then $(A, N)$ is an L. I. pair.

**Proof.** Let $u \in A$ be such that $u^2 - u \in N$. We shall produce an idempotent $e \in N$ such that $u - e \in N$.

Since $R_x$ is a flat $R$-module, without ambiguity we can let $N_x$ denote the image of $N$ under the canonical homomorphism $A \to A/I(x)A = A_x$. Now $u_x = u + I(x)A$ is an element of $A_x$ such that $u_x + N_x$ is an idempotent element of $A_x/N_x$. Since $(A_x, N_x)$ is an L. I. pair and since an idempotent in $A_x$ can be lifted to an idempotent in $A$ [11, Proposition II.20, p. 34], there exists an idempotent $f(x) \in A$ such that $u_x = [f(x)]_x + [n(x)]_x$ for some $n(x) \in N$. By [11, Proposition II.16, p. 32] for each $x \in X(R)$ there exists an idempotent $e(x) \in R$ such that $u_e(x) = f(x)e(x) + n(x)e(x)$ and $u_y = [f(x)]_y + [h(x)]_y$ for all $y \in N(e(x))$.

By the "partition property" of $X(R)$, the open cover $\{N(e(x))\}$ of $X(R)$ has a finite refinement of disjoint, open, closed sets $\{N(e_i)\}_{i=1}^m$, for some idempotents $e_i \in R$, which covers $X(R)$. Since $\{N(e_i)\}_{i=1}^m$ is a cover of $X(R)$ by disjoint sets, it follows that $1 = \sum_{i=1}^m e_i$ and $e_i e_j = 0$ for each $i \neq j$. Furthermore $N(e_i) \subseteq N(e(x_i))$ for some $x_i \in X(R)$, and thus it can be shown that $e_i e(x_{i,j}) = e_{i,j}$; hence, $u e_i = f_{i} e_i + n_{i} e_i$ where $f_i = f(x_i)$ and $n_i = n(x_i)$. Since the $e_i$ are pairwise orthogonal idempotents and each $f_i$ is an idempotent, then $e = \sum_{i=1}^m f_i e_i$ is an idempotent. Now

$$u = \sum_{i=1}^m u e_i = \sum_{i=1}^m (f_i e_i + n_i e_i) = e + \sum_{i=1}^m n_i e_i,$$

and therefore $u - e \in N$.

**Corollary 3.** If $R$ is an inertial coefficient ring then $(R, \text{rad}(R))$ is an L. I. pair.

**Proof.** Each stalk $R_x$ is a connected ring and is an inertial coefficient ring because it is a homomorphic image of $R$. By Theorem 1, $(R_x, \text{rad}(R_x))$ is an L. I. pair for every $x \in X(R)$. Since $(\text{rad } R)_x \subseteq \text{rad}(R_x)$ and $R_x$ is a commutative ring, then $(R_x, (\text{rad } R)_x)$ is an L. I. pair for every $x \in X(R)$ [5,
Corollary 1.3, p. 46]; hence by Proposition 2, \((R, \text{rad}(R))\) is an L. I. pair.

We are now able to prove the general case:

**Theorem 4.** Let \(R\) be an inertial coefficient ring and \(A\) a finitely generated \(R\)-algebra. Then \((A, N)\) is an L. I. pair.

**Proof.** By replacing \(R/\text{annih}_RA\) by \(R\) we may assume that \(A\) is a faithful \(R\)-algebra.

Let \(u \in A\) be such that \(u^2 - u \in N\). We must show there exists an idempotent \(e \in A\) such that \(u - e \in N\). Let \(B\) denote the \(R\)-subalgebra of \(A\) generated by \(u\). Now \(B/B \cap N\), being a homomorphic image of \((R/\text{rad}(R))[x]/\langle x^2 - x \rangle\), is a separable \(R\)-algebra. Since \(B\) is a finitely generated \(R\)-algebra, \(B \cap N \subseteq \text{rad}(B)\) [1, Corollary, p. 126], and hence \(B/\text{rad}(B)\) is a separable \(R\)-algebra. Since \(B\) is a commutative \(R\)-algebra, by [7, see proof of Proposition 3.3, p. 85] \(B\) is an inertial coefficient ring, and thus by Corollary 3 \((B, \text{rad}(B))\) is an L. I. pair. It follows from the commutativity of \(B\) that \((B, N \cap B)\) is an L. I. pair. Then \(u^2 - u \in N \cap B\) implies that there exists an idempotent \(e \in B\) such that \(u - e \in N \cap B\). Then \(e \in A\) and \(u - e \in N\).

**Corollary 5.** An inertial coefficient ring is a Hensel ring.

**Proof.** To show \(R\) is a Hensel ring it suffices to show that for any finitely generated, commutative, free \(R\)-algebra \(A\), \((A, \text{rad}(R)A)\) is an L. I. pair [5, Theorem 4.1, p. 55]. Since \(R\) is an inertial coefficient ring, \((A, N)\) is an L. I. pair, and since \(\text{rad}(R)A \subseteq N\) and \(A\) is commutative, then \((A, \text{rad}(R)A)\) is an L. I. pair.

The following corollary to Theorem 4 follows immediately from [8, Corollary 2, p. 556].

**Corollary 6.** If \(R\) is an inertial coefficient ring and if \(S\) and \(T\) are two inertial subalgebras of a finitely generated \(R\)-algebra \(A\), then \(T = (1 + n)^{-1} \cdot S(1 + n)\) for some \(n \in N\).

N. S. Ford has given an example of a ring \(R\) and an \(R\)-algebra \(A\) such that \(S\) and \(T\) are not isomorphic [4].

The following corollary to Theorem 4 follows immediately from [8, Theorem 2, p. 554].

**Corollary 7.** The following are equivalent properties of a commutative ring \(R\):

(i) For all finitely generated, commutative \(R\)-algebras \(A\) such that \(A/N\) is \(R\)-separable, there exists a separable \(R\)-subalgebra \(S\) of \(A\) such that \(A = S + N\).

(ii) All finitely generated, commutative \(R\)-algebras have the idempotent lifting property.

If Ingraham's conjecture is true then it is unnecessary to restrict the
algebras $A$ in Corollary 7 to commutative $R$-algebras, for then the lifting of idempotents from $A/N$ to $A$ in all finitely generated $R$-algebras $A$ is equivalent to the lifting of the separability of $A/N$ to a separable subalgebra $S$ of $A$ in all finitely generated $R$-algebras.

**References**


Department of Mathematics, Wake Forest University, Winston-Salem, North Carolina 27109