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THE CONJUGACY PROBLEM FOR *HNN* GROUPS AND THE WORD PROBLEM FOR COMMUTATIVE SEMIGROUPS

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ABSTRACT. We show how a positive solution to the word problem for finitely generated commutative semigroups leads to a positive solution to the conjugacy problem for a class of *HNN* groups.

Here we answer a question of Professor Wilhelm Magnus concerning the solvability of the conjugacy problem of the groups VA discussed in [1]. These groups are given by

$$\text{(I)} \qquad \langle a_1, \ldots, a_k, b;\ a_1^{-1} b^{p_1} a_1 = b^{q_1}, \ldots, a_k^{-1} b^{p_k} a_k = b^{q_k} \rangle$$

where $p_i q_i \geqslant 1$ and $(p_i, q_i) = 1$. We call the groups given by (I) *VA-groups* and denote them by $G(p_1, q_1, \ldots, p_k, q_k)$ as in [1]. We call the integers appearing in (I) the *exponents* of the group. Let l and m be nonzero integers and call m *reachable* from l with respect to the exponents if there is a sequence of integers beginning with l and ending with m, such that for successive terms l_1 and l_{i+1} either $l_{i+1} = l_i(q_j/p_j)$ or $l_{i+1} = l_i(p_j/q_j)$. The *reachability problem* for the exponents is to decide for arbitrary such l and m whether m is reachable from l.

From [1, Lemma 1] we obtain

LEMMA 1. *A VA-group has solvable conjugacy problem if and only if the reachability problem for its exponents is solvable.*

The reachability problem for the exponents of VA-group is equivalent to the reachability problem for a class of self-dual vector addition systems [1]. Recently several computer scientists have solved the reachability problem for self-dual vector addition systems using combinatorial methods. Below we give a simple self-contained algebraic proof of the solvability of the reachability problem for the exponents. This allows us to prove

THEOREM. *VA-groups have solvable conjugacy problem.*

It suffices to consider the reachability problem restricted to the set M of positive integers whose prime divisors are among the prime divisors of the exponents $p_1, q_1, \ldots, p_k, q_k$. Observe that M forms a multiplicative semigroup and the reachability relation induces a congruence on M. Denote the

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finitely generated commutative semigroup resulting from this congruence by $S(p_1, q_1, \ldots, p_k, q_k)$.

We may construct a presentation of $S(p_1, q_1, \ldots, p_k, q_k)$ in the following manner. Let $d_0, \ldots, d_n$ denote the positive prime divisors of the exponents. Let p_i and q_i have prime decomposition

$$p_i = d_0^{e_{0i}} \cdots d_n^{e_{ni}}, \qquad q_i = d_0^{e'_{0i}} \cdots d_n^{e'_{ni}}$$

where $e_{ji}, e'_{ji} \geq 0$ for $j = 0, \ldots, n$. We associate a generating symbol x_i with d_i for $i = 0, \ldots, k$. The exponents p_i, q_i are seen to be coded, respectively, as the words C_i, D_i where

$$C_i = x_0^{e_{0i}} \cdots x_n^{e_{ni}}, \qquad D_i = x_0^{e'_{0i}} \cdots x_n^{e'_{ni}}.$$

Then $S(p_1, q_1, \ldots, p_k, q_k)$ has the following presentation:

$$\langle x_0, \ldots, x_n; x_u x_v = x_v x_u, C_i = D_i, 0 \leq u, v \leq n, 1 \leq i \leq k \rangle.$$

Hence, we have proved

LEMMA 2. *The reachability problem for the exponents of $G(p_i, q_i, \ldots, p_k, q_k)$ is reducible to the word problem for the finitely presented commutative semigroup $S(p_1, q_1, \ldots, p_k, q_k)$.*

It has been pointed out to the author by Gilbert Baumslag and George Bergman that finitely generated commutative semigroups have solvable word problem. Baumslag observes that a finitely generated commutative semigroup S is embedded in the integral semigroup ring R formed from S. R is a finitely generated commutative ring and is therefore residually finite [3, pp. 64–65] from which it follows that S is residually finite. By a theorem of L. Rédei [4], S is finitely related and, hence, finitely presented. It is well known that finitely presented residually finite algebras have solvable word problem (a result which is attributed by some workers in the field to Verena Dyson). Hence S has solvable word problem. Our Theorem is now immediate from Lemmas 1 and 2 and the preceding remark.

Generalizations to the *HNN* groups considered in [2] are straightforward.

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