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LEFT LOOPS WHICH SATISFY THE LEFT BOL IDENTITY

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ABSTRACT. It is our purpose in this paper to initiate a study of the algebraic properties of a left loop $Q(\cdot)$ satisfying the identical relation

$$y(z \cdot yx) = (y \cdot zy)x \tag{1}$$

for all $x, y, z \in Q$. It is shown that (1) implies right division in $Q(\cdot)$. By introducing a new operation ' $\circ$ ' in Q, the connection between the left loop $Q(\cdot)$ and Bol loop $Q(\circ)$ is established. Further we show that the role of nuclei in the left loop theory is not the same as that in the loop theory. We conclude the paper by describing situations in which the left loop $Q(\cdot)$ is Moufang.

1. **Introduction.** A loop $Q(\cdot)$ is a Bol loop if and only if

$$(xy \cdot z)y = x(yz \cdot y) \tag{1}$$

for all $x, y, z \in Q$. Bol loops had their origin in the work of Bol [1] and their algebraic properties have been discussed by Robinson [2]. There exists a duality between those loops satisfying (1) and those loops for which

$$y(z \cdot yx) = (y \cdot zy)x \tag{2}$$

for all $x, y, z \in Q$. (1) and (2) are called right Bol identity and left Bol identity respectively. A loop for which (1) ((2)) holds is called a loop with R.B.P. (L.B.P.).

In this paper we discuss the algebraic properties of a left loop $Q(\cdot)$ with L.B.P. We show that (2) implies right division in $Q(\cdot)$. Secondly we introduce a new operation ' $\circ$ ' in Q and prove that $Q(\circ)$ is a loop with L.B.P. Robinson [2] has proved results for the loops with R.B.P., so similar theorems hold for the loop $Q(\circ)$ and this fact helps us in discussing results for $Q(\cdot)$. Further we emphasize some principal distinction between the left loop theory and the loop theory. Namely, such notions as nuclei are not valid in the left loop theory. Instead, some new notions introduced are more effective in the case of a left loop with L.B.P. We conclude the paper by describing situations in which a

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left loop with L.B.P. is Moufang. We recall that a left quasi-group is a magma (groupoid) $Q(\cdot)$ in which the equation $a \cdot x = b$ is uniquely soluble: $x = a \backslash b$. A left quasi-group $Q(\cdot)$ with right identity ε $(a \cdot \varepsilon = a, \forall a \in Q)$ is called a left loop.

In this paper we denote the right inverse of each $x \in Q$ by x^ρ, i.e., $x \cdot x^\rho = \varepsilon$.

2. THEOREM 1. *If $Q(\cdot)$ is a left loop with L.B.P., then*
(i) *$Q(\cdot)$ satisfies the left inverse property,*
(ii) *$(x \cdot yx)^\rho = x^\rho \cdot y^\rho x^\rho$ for all $x, y \in Q$.*

PROOF. (i) Let $y = z^\rho$ in (2); then $z^\rho(z \cdot z^\rho x) = (z^\rho \cdot zz^\rho)x = z^\rho x$ for all $x, z \in Q$. Thus, by the left cancellative law, we have

(3) $$z \cdot z^\rho x = x \quad \text{for all } x, z \in Q.$$

Let $z = y^\rho$ in (2); then $y(y^\rho \cdot yx) = (y \cdot y^\rho y)x = yx$ by (3), for all $x, y \in Q$. Thus, by the left cancellative law, we have

(4) $$z^\rho \cdot zx = x \quad \text{for all } x, z \in Q.$$

This completes the proof of (i). Let $x = \varepsilon$ in (4); then $z^\rho \cdot z = \varepsilon$ or

(5) $$(z^\rho)^\rho = z \quad \text{for all } z \in Q.$$

Further, $\varepsilon \cdot \varepsilon = \varepsilon \Rightarrow \varepsilon^\rho = \varepsilon$. Let $z = \varepsilon$ in (4); then

(6) $$\varepsilon \cdot \varepsilon x = x \quad \text{for all } x \in Q.$$

(ii) For all $x, y \in Q$,

$$\begin{aligned}(x^\rho \cdot y^\rho x^\rho)(x \cdot yx) &= x^\rho[y^\rho(x^\rho(x \cdot yx))] \quad \text{by (2)} \\ &= x^\rho(y^\rho \cdot yx) \quad \text{by (4)} \\ &= x^\rho x \quad \text{by (4)} \\ &= \varepsilon.\end{aligned}$$

So, $(x \cdot yx)^\rho = x^\rho \cdot y^\rho x^\rho$ for all $x, y \in Q$. This completes the proof of (ii).

THEOREM 2. *If $Q(\cdot)$ is a left loop with L.B.P., then for given $a, b \in Q$, the equation $x \cdot a = b$ has a unique solution for x in $Q(\cdot)$.*

PROOF. Let $x = y^\rho$ in (2); then

(7) $$y \cdot z = (y \cdot zy)y^\rho \quad \text{for all } y, z \in Q.$$

Let us consider the equation

(8) $$x \cdot a = b \quad \text{for given } a, b \in Q.$$

If a solution for x exists, then

$$\begin{aligned} a \cdot x &= (a \cdot xa)a^{\rho} \quad \text{by (7)} \\ &= ab \cdot a^{\rho} \quad \text{by (8)} . \end{aligned}$$

Thus $x = a\backslash ab \cdot a^{\rho} = a^{\rho}(ab \cdot a^{\rho})$. Substituting the value of x in (8), we have

$$\begin{aligned} [a^{\rho}(ab \cdot a^{\rho})]a &= a^{\rho} \cdot ab \quad \text{by (7)} \\ &= b \quad \text{by (4).} \end{aligned}$$

Thus $x \cdot a = b$ has a unique solution for x in $Q(\cdot)$.

We define a new operation ' $\circ$ ' in the left loop $Q(\cdot)$ as follows.

$$x \circ y =_{\text{def}} x \cdot L_{\varepsilon}^{-1} y \tag{9}$$

where $L_x y := xy$.

LEMMA 1. *The operator L_{ε} is an involutive automorphism in the left loop $Q(\cdot)$ with L.B.P.*

PROOF. By equation (6), $\varepsilon \cdot \varepsilon x = x$ for all $x \in Q$. Therefore, $L_{\varepsilon}(L_{\varepsilon} x) = x$, $(L_{\varepsilon} L_{\varepsilon})(x) = x$ implies

$$L_{\varepsilon} = L_{\varepsilon}^{-1} . \tag{10}$$

Let $y = \varepsilon$ in (2); then $\varepsilon(z \cdot \varepsilon x) = \varepsilon z \cdot x$ for all $x, z \in Q$. Thus

$$L_{\varepsilon}(z \cdot L_{\varepsilon} x) = L_{\varepsilon} z \cdot x. \tag{11}$$

Putting $w = L_{\varepsilon} x \Rightarrow x = L_{\varepsilon}^{-1} w = L_{\varepsilon} w$ in (11), we have

$$L_{\varepsilon}(z \cdot w) = (L_{\varepsilon} z) \cdot (L_{\varepsilon} w) \quad \text{for all } z, w \in Q. \tag{12}$$

This completes the proof of Lemma 1.

COROLLARY 1. *The operator L_{ε} is an involutive automorphism of $Q(\circ)$.*

PROOF. We consider

$$\begin{aligned} L_{\varepsilon}(x \circ y) &= L_{\varepsilon}(x \cdot \varepsilon y) \quad \text{by (9) and (10)} \\ &= (L_{\varepsilon} x) \cdot y \quad \text{by (12) and (6)} \\ &= (L_{\varepsilon} x) \circ (\varepsilon y) \\ &= (L_{\varepsilon} x) \circ (L_{\varepsilon} y) \quad \text{for all } x, y \in Q. \end{aligned}$$

This completes the proof of Corollary 1.

COROLLARY 2. *The operator L_{ε} satisfies the property*

$$L_{\varepsilon}(x^{\rho}) = (L_{\varepsilon} x)^{\rho} \quad \textit{for all } x \in Q. \tag{13}$$

PROOF. For all $x \in Q$, $x \cdot x^\rho = \varepsilon$; thus $L_\varepsilon(x \cdot x^\rho) = L_\varepsilon(\varepsilon)$. By using (9) and (12), we have

$$(L_\varepsilon x) \cdot (L_\varepsilon x^\rho) = \varepsilon \Rightarrow L_\varepsilon(x^\rho) = (L_\varepsilon x)^\rho.$$

This completes the proof of Corollary 2.

THEOREM 3. *$Q(\circ)$ is a loop with L.B.P. and having ε as two-sided identity.*

PROOF. By definition (9), for all $x, y \in Q$, $x \circ y = x \cdot \varepsilon y = z \in Q(\circ)$. Consider the equation $a \circ x = b$ for all $a, b \in Q$, $a \circ x = a \cdot \varepsilon x = b \Rightarrow \varepsilon x = a \backslash b$ or $x = \varepsilon \cdot a^\rho b \Rightarrow Q(\circ)$ has a unique left division. Now we consider the equation $x \circ a = b$ for all $a, b \in Q$, $x \circ a = x \cdot \varepsilon a = b \Rightarrow x = b \cdot (\varepsilon a)^\rho$ by Theorem 2 $= b \cdot \varepsilon a^\rho$ by (13) $\Rightarrow Q(\circ)$ has a unique right division. Next we consider, for all $x \in Q$, $\varepsilon \circ x = \varepsilon \cdot \varepsilon x = x$ by (6), also $x \circ \varepsilon = x \cdot \varepsilon\varepsilon = x$. Thus ε acts as two-sided identity. Thus we have proved that $Q(\circ)$ is a loop. Next we prove that $Q(\circ)$ has L.B.P.

For all $x, y, z \in Q$,

$$\begin{aligned} y \circ (z \circ (y \circ x)) &= y(\varepsilon(z(\varepsilon(y \cdot \varepsilon x)))) = y(\varepsilon(z(y \cdot \varepsilon x))) \\ &= (y(\varepsilon z \cdot y)) \cdot (\varepsilon x) = (y(\varepsilon(z \cdot \varepsilon y))) \cdot (\varepsilon x) \\ &= (y \circ (z \circ y)) \circ x. \end{aligned}$$

Thus,

$$y \circ (z \circ (y \circ x)) = (y \circ (z \circ y)) \circ x \quad \text{for all } x, y, z \in Q.$$

Hence the loop $Q(\circ)$ has L.B.P.

THEOREM 4. *If $Q(\circ)$ is a loop with L.B.P. and L is an arbitrary involutive automorphism of $Q(\circ)$, then $Q(\cdot)$ equipped with the multiplication $x \cdot y =_{\text{def}} x \circ Ly$ is a left loop with L.B.P. and L is an involutive automorphism of $Q(\cdot)$.*

PROOF. Since L is an involutive automorphism of $Q(\circ)$ and

$$x \cdot y =_{\text{def}} x \circ Ly \quad \text{for all } x, y \in Q, \tag{14}$$

there exists left division and right division in $Q(\cdot)$. Let $x = e$ (identity of $Q(\circ)$) in (14); then

$$Ly = e \cdot y \quad \text{for all } y \in Q. \tag{15}$$

If we put $y = e$ in (15), we have

$$Le = e. \tag{16}$$

Since L is an involutive automorphism of $Q(\circ)$, we have $L = L^{-1}$. Secondly,

$$L(x \cdot y) = L(x \circ Ly) = (Lx) \circ L(Ly) = (Lx) \circ y = (Lx) \cdot (Ly).$$

Thus,

$$L(x \cdot y) = Lx \cdot Ly \quad \text{for all } x, y \in Q.$$

We have shown that L is an involutive automorphism of $Q(\cdot)$. Now we consider

$$\begin{aligned} y(z \cdot yx) &= y \circ L(z \circ L(y \circ Lx)) = y \circ (Lz \circ (y \circ Lx)) \\ &= (y \circ (Lz \circ y)) \circ Lx \quad \text{by (2)} \\ &= (y \circ L(z \circ Ly)) \circ Lx = (y \cdot zy)x. \end{aligned}$$

Thus $Q(\cdot)$ satisfies the L.B.P. This completes the proof of the theorem.

THEOREM 5. *If $Q(\circ)$ is a loop with identity e and L.B.P., and $Q(\cdot)$ is a left loop with right identity ε and L.B.P., such that $x \circ Ly = x \cdot y$ and $e = \varepsilon$, then L is an involutive automorphism of $Q(\circ)$ and $Q(\cdot)$.*

PROOF. For all $x, y \in Q$,

$$x \circ Ly = x \cdot y. \tag{17}$$

Let $x = e = \varepsilon$ in (17); then

$$Ly = \varepsilon \cdot y \quad \text{for all } y \in Q. \tag{18}$$

Since $Q(\cdot)$ is a left loop with L.B.P. and by Lemma 1, it follows that L is an involutive automorphism of $Q(\cdot)$. Now (17) can be written as follows:

$$x \circ y = x \cdot Ly \quad \text{for all } x, y \in Q. \tag{19}$$

By Corollary 1 of Lemma 1 it follows that L is an involutive automorphism of $Q(\circ)$. This completes the proof of the theorem.

Following Bruck [3, p. 250] the left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop $G(\cdot)$ are defined by:

$$N_\lambda(\cdot) = \{\text{all } x \in G \,|\, x \cdot yz = xy \cdot z, \text{ all } y, z \in G\}, \tag{20}$$

$$N_\mu(\cdot) = \{\text{all } y \in G \,|\, x \cdot yz = xy \cdot z, \text{ all } x, z \in G\}, \tag{21}$$

$$N_\rho(\cdot) = \{\text{all } z \in G \,|\, x \cdot yz = xy \cdot z, \text{ all } x, y \in G\}. \tag{22}$$

Further, Bruck [3, Theorem 2A, p. 255] has proved that each subset $N_\lambda(\cdot)$, $N_\mu(\cdot)$, $N_\rho(\cdot)$ is an associative subloop (group) of $G(\cdot)$. Thus each subset $N_\lambda(\circ)$, $N_\mu(\circ)$ and $N_\rho(\circ)$ of $Q(\circ)$ is an associative subloop of $Q(\circ)$. But we cannot assert for example, that $N_\lambda(\cdot)$ is an associative subloop for any left loop $Q(\cdot)$. This fact suggests the following theorem for the left loop $Q(\cdot)$.

THEOREM 6. *If $Q(\cdot)$ is a left loop with L.B.P., then there exist sub-left loops*
(a) $Q_\lambda(\cdot) = \{\text{all } a \in Q | ax \cdot \varepsilon y = a \cdot xy \text{ for all } x, y \in Q\}$,
(b) $Q_\mu(\cdot) = \{\text{all } a \in Q | xa \cdot \varepsilon y = x \cdot ay \text{ for all } x, y \in Q\}$,
(c) $Q_\rho(\cdot) = \{\text{all } a \in Q | xy \cdot \varepsilon a = x \cdot ya \text{ for all } x, y \in Q\}$,
of $Q(\cdot)$ such that $N_\lambda(\cdot) \cap Q_\lambda(\cdot) = \varnothing$, $N_\mu(\cdot) \cap Q_\mu(\cdot) = \varnothing$ and $N_\rho(\cdot) \subset Q_\rho(\cdot)$, where $\varnothing$ denotes the empty set.

PROOF. If a, b are two elements of $Q_\lambda(\cdot)$ and x, y any two elements of $Q(\cdot)$, then

$$\begin{aligned} ab \cdot xy &= ab \cdot \varepsilon(\varepsilon x \cdot \varepsilon y) = a(b(\varepsilon \cdot xy)) \quad \text{by (a)} \\ &= a(b(\varepsilon x \cdot \varepsilon y)) = a((b \cdot \varepsilon x)y) \\ &= (a(b \cdot \varepsilon x)) \cdot (\varepsilon y) = (ab \cdot x) \cdot (\varepsilon y). \end{aligned}$$

Thus $a \cdot b \in Q_\lambda(\cdot)$. Also

$$\begin{aligned} a(a^\rho x \cdot y) &= (a \cdot a^\rho x) \cdot (\varepsilon y) = x \cdot \varepsilon y \quad \text{by (3)} \\ &= \varepsilon(\varepsilon x \cdot y) = (aa^\rho) \cdot (\varepsilon x \cdot y) = a(a^\rho(\varepsilon(\varepsilon x \cdot y))) = a(a^\rho(x \cdot \varepsilon y)); \end{aligned}$$

by the left cancellative law $a^\rho x \cdot y = a^\rho(x \cdot \varepsilon y) \Rightarrow a^\rho \in Q_\lambda(\cdot)$. Thus $Q_\lambda(\cdot)$ is a sub-left loop of $Q(\cdot)$. Let $a \in Q_\lambda(\cdot) \cap N_\lambda(\cdot)$; then $a \cdot xy = ax \cdot y$ and $ax \cdot \varepsilon y = a \cdot xy$ for all $x, y \in Q$. Thus we have $ax \cdot y = ax \cdot \varepsilon y$ by the left cancellative law, $y = \varepsilon y$ for all $y \in Q$. It is a contradiction and hence $N_\lambda(\cdot) \cap Q_\lambda(\cdot) = \varnothing$.

Similarly we can prove the results for $N_\mu(\cdot)$ and $Q_\mu(\cdot)$. From (22) and (c), we get $N_\rho(\cdot) = \{\text{all } a \in Q_\rho(\cdot) | \varepsilon \cdot a = a \text{ for all } a \in Q_\rho(\cdot)\}$. It implies that $N_\rho(\cdot) \subset Q_\rho(\cdot)$.

THEOREM 7. *If $Q(\cdot)$ is a left loop with L.B.P. and satisfies the identity $(x \cdot y)^\rho = y^\rho \cdot x^\rho$ for all $x, y \in Q$, then $Q(\cdot)$ is a Moufang loop.*

PROOF. $(x \cdot y)^\rho = y^\rho \cdot x^\rho \Rightarrow (\varepsilon \cdot y)^\rho = y^\rho$. But $(\varepsilon \cdot y)^\rho = \varepsilon \cdot y^\rho$. Consequently $\varepsilon \cdot y^\rho = y^\rho$ for all $y \in Q$ which implies $\varepsilon \cdot z = z$ for all $z \in Q$, and the right identity ε is left as well. From Theorem 1(ii),

$$\begin{aligned} x^\rho \cdot y^\rho x^\rho &= (x \cdot yx)^\rho = (yx)^\rho \cdot x^\rho = x^\rho y^\rho \cdot x^\rho \\ &\Rightarrow x \cdot yx = xy \cdot x \quad \text{for all } x, y \in Q. \end{aligned} \tag{23}$$

For all $x, y, z \in Q$,

$$\begin{aligned} x(y \cdot xz) &= (x \cdot yx)z \quad \text{by (2)} \\ &= (xy \cdot x)z \quad \text{by (23)}. \end{aligned}$$

Thus $Q(\cdot)$ satisfies the Moufang identity. Hence, by Bruck's Lemma [4, Lemma 3.1, p. 115], $Q(\cdot)$ is a Moufang loop.

COROLLARY 3. *If $Q(\cdot)$ is a left loop with L.B.P. and diassociative, then it is a Moufang loop.*

The proof is obvious from Theorem 7.

THEOREM 8. *If $Q(\cdot)$ is a left loop with L.B.P. and is right alternative, then it is a Moufang loop.*

PROOF. Since $Q(\cdot)$ is right alternative, $xy \cdot y = x \cdot yy$ for all $x, y \in Q$. Therefore, $y(xy \cdot y) = y(x \cdot yy) = (y \cdot xy)y$ by (2). Put $xy = z$. Thus $y \cdot zy = yz \cdot y$ for all $y, z \in Q$. The rest of the proof follows from Theorem 7.

THEOREM 9. *If $Q(\cdot)$ is a left loop with L.B.P. and R.B.P., then it is a Moufang loop.*

PROOF. Since $Q(\cdot)$ satisfies (1) and by putting $z = \varepsilon$ in (1), we have

$$xy \cdot y = x \cdot yy \quad \text{for all } x, y \in Q.$$

Thus $Q(\cdot)$ is right alternative. Hence $Q(\cdot)$ is a Moufang loop by Theorem 8.

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