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$L_p(\mu, X)(1 < p < \infty)$ HAS THE RADON-NIKODÝM PROPERTY IF X DOES BY MARTINGALES

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ABSTRACT. Using the fact that $L_p[0, 1]$ $(1 < p < \infty)$ has an unconditional basis, Sundaresan has shown that $L_p(\mu, X)$ has the Radon-Nikodým property if $1 < p < \infty$ and X has the Radon-Nikodým property. In this note, Sundaresan's theorem is proved by direct martingale methods. Then it is shown how to adapt this argument to the context of Orlicz spaces in which Sundaresan's argument is not applicable.

Using the fact that $L_p([0, 1])$ $(1 < p < \infty)$ has an unconditional basis, Sundaresan [7] has shown that if X is a Banach space and $(S, \mathfrak{F}, \lambda)$ is a finite measure space, then $L_p(S, \mathfrak{F}, \lambda, X)$ $(1 < \mathrm{p} < \infty)$ has the Radon-Nikodým property. Since unconditional bases of $L_p[0, 1]$ are closely related to martingale difference sequences (see Burkholder [1] and Dor and Odell [3]), it is irresistably tempting to prove Sundaresan's result using a direct martingale argument. This note is the result of yielding to that temptation.

THEOREM 1 (SUNDARESAN [7]). *Let $(S, \mathfrak{F}, \lambda)$ be a finite measure space and $1 < p < \infty$. If X is a Banach space with the Radon-Nikodým property, then $L_p(S, \Sigma, \lambda, X)$ also has the Radon-Nikodým property.*

PROOF. Let (Ω, Σ, μ) be a finite measure space and $F\colon \Sigma \to L_p(S, \mathfrak{F}, \lambda, X)$ be a μ-continuous vector measure of bounded variation. There is no loss of generality in assuming that $\|F(E)\|_{L_p(\lambda, X)} \leqslant \mu(E)$ for all $E \in \Sigma$. Now let π be a partition of Ω into a finite number of sets in Σ and Δ be a partition of S into a finite number of sets in $\mathfrak{F}$ and define

$$f_{\pi,\Delta}(s, t) = \sum_{E \in \pi} \sum_{I \in \Delta} \frac{\int_I F(E)\,d\lambda}{\mu(E)\lambda(I)} \chi_I(s)\chi_E(t)$$

for $(s, t) \in S \times \Omega$. (Hence $0/0 = 0$.) Since the X-valued set function $\int_I F(E)\,d\lambda$ is finitely additive in both $I \in \mathfrak{F}$ and $E \in \Sigma$, it is clear that $(f_{\pi,\Delta}; B_{\pi,\Delta})$ is a martingale in $L_p(\lambda \times \mu, X)$ (here $B_{\pi,\Delta}$ is the trivial σ-field generated by sets of the form $E \times I$ with $E \in \pi$ and $I \in \Delta$). Since X has the Radon-Nikodým property, the martingale $(f_{\pi,\Delta}; B_{\pi,\Delta})$ is $L_p(\lambda \times \mu, X)$ convergent if it is $L_p(\lambda \times \mu, X)$-bounded [2]. Now

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$$\|f_{\pi,\Delta}\|^p_{L_p(\lambda\times\mu,X)} = \sum_{E\in\pi}\sum_{I\in\Delta} \frac{\|\int_I F(E)\,d\lambda\|^p_X}{\mu(E)^p\lambda(I)^p}\,\mu(E)\lambda(I).$$

But

$$\left\|\int_I F(E)\,d\lambda\right\|^p_X = \left\|\int_I F(E)\chi_I\,d\lambda\right\|^p_X$$
$$<\|F(E)\chi_I\|^p_{L_p(\lambda,X)}\lambda(I)^{p/q} \qquad (p^{-1}+q^{-1}=1)$$

by the Hölder inequality. Thus

$$\|f_{\pi,\Delta}\|^p_{L_p(\lambda\times\mu,X)} < \sum_{E\in\pi}\sum_{I\in\Delta}\frac{\|F(E)\chi_I\|^p_{L_p(\lambda,X)}}{\mu(E)^p}\,\lambda(I)^{1+p/q-p}\mu(E)$$
$$= \sum_{E\in\pi}\sum_{I\in\Delta}\frac{\|F(E)\chi_I\|^p_{L_p(\lambda,X)}}{\mu(E)^p}\,\mu(E)$$

since $1 + p/q - p = 0$. Now note that $\|F(E)\chi_I\|^p_{L_p(\lambda,X)}$ is an additive function of $I \in \Delta$. Hence

$$\|f_{\pi,\Delta}\|^p_{L_p(\lambda\times\mu,X)} \leqslant \sum_{E\in\pi}\frac{\|F(E)\|^p_{L_p(\lambda,X)}\mu(E)}{\mu(E)^p}$$
$$\leqslant \sum_{E\in\pi}\mu(E) = \mu(\Omega)$$

since $\|F(E)\|^p_{L_p(\lambda,X)} < \mu(E)$ for all $E\in\Sigma$. Accordingly,

$$\lim_{\pi,\Delta} f_{\pi,\Delta} = f \in L_p(\lambda\times\mu, X)$$

exists in $L_p(\lambda\times\mu, X)$ norm. Now note that

$$\int \|f(s,t)\|^p_X\,d\lambda(s)\,d\mu(t) < \infty.$$

Hence $f(\cdot,t)\in L_p(\lambda, X)$ for μ-almost all $t\in\Omega$. Redefine f to be zero on the exceptional set. Set $g(t) = f(\cdot,t)$ for $t\in\Omega$. Then g is an $L_p(\lambda, X)$-valued μ-Bochner integrable function (see Dunford and Schwartz [5, III.11.16]). Finally if $E_1\in\Sigma$

$$\int_{E_1} g\,d\mu = \lim_{\pi,\Delta}\int_{E_1}\sum_{E\in\pi}\sum_{I\in\Delta}\frac{\int_I F(E)\,d\lambda}{\mu(E)\lambda(I)}\,\chi_E\chi_I\,d\mu$$
$$= \lim_{\pi}\int_{E_1}\sum_{E\in\pi}\left(\lim_{\Delta}\sum_{I\in\Delta}\frac{\int_I F(E)\,d\lambda}{\lambda(I)}\,\chi_I\right)\frac{\chi_E}{\mu(E)}\,d\mu$$
$$= \lim_{\pi}\int_{E_1}\sum_{E\in\pi}\frac{F(E)}{\mu(E)}\,\chi_E\,d\mu$$

since $\Sigma_{I\in\Delta}(\int_I F(E)d\lambda/\lambda(I))\chi_I$ is a martinglae in $L_p(\lambda, X)$ converging to $F(E)$ in $L_p(\lambda, X)$-norm. But since

$$\lim_\pi \int_{E_1} \sum_{E\in\pi} \frac{F(E)}{\mu(E)} \chi_E d\mu = F(E_1),$$

$F(E_1) = \int_{E_1} g d\mu$ for all $E_1 \in \Sigma$, as required.

Sundaresan [7] has noted that $L_\Phi(\mu, X)$ has the Radon-Nikodým property if X has the Radon-Nikodým property and L_Φ [0, 1] has an unconditional basis. According to Gapoškin [6], separable nonreflexive Orlicz spaces fail to have an unconditional basis. In this section, we shall indicate how the arguments used in the first section extend to the widest possible class of Orlicz spaces, a class including separable nonreflexive Orlicz spaces.

THEOREM 2. *Let* $(S, \mathfrak{F}, \lambda)$ *be a fintie measure space and* Φ *be a Young's function (i.e. Orlicz function) such that*

(i) $\lim_{t\to\infty}\Phi(t)/t = \infty$ *and*

(ii) *there exist constants* K *and* M *such* $\Phi(2t) \leqslant K\Phi(t)$ *for* $x \geqslant M$ *(i.e.* Φ *satisfies the* Δ_2*-condition for large* t*). If* X *is a Banach space with the Radon-Nikodým property, then* $L_\Phi(\lambda,X)$ *also has the Radon-Nikodým property.*

PROOF. Only an outline of the proof will be given since the proof is very similar to the proof of Theorem 1.

The basic facts needed are as follows. By (ii), simple functions are dense in $L_\Phi(\mu, Y)$ for any finite measure μ and any Banach space Y. Consequently, an $L_1(\mu, Y)$ convergent martingale that is bounded in $L_\Phi(\mu, Y)$ also converges in $L_\Phi(\mu, Y)$ norm [9].

Let (Ω, Σ, μ) be a finite measure space and $F: \Sigma \to L_\Phi(\lambda, X)$ be a vector measure with $\|F(E)\|_{L_\Phi(\mu,X)} \leqslant \mu(E)$ for all $E \in \Sigma$. As in the L_p case, let

$$f_{\pi,\Delta} = \sum_{E\in\pi}\sum_{I\in\Delta} \frac{\int_I F(E)d\lambda}{\mu(E)\lambda(I)}\chi_E\chi_I.$$

Then $(f_{\pi,\Delta})$ is a martingale in $L_\Phi(\mu\times\lambda, X)$. Moreover,

$$\int_{\Omega\times S}\Phi(\|f_{\pi,\Delta}\|_X)d\lambda d\mu$$

$$= \sum_{E\in\pi}\mu(E)\sum_{I\in\Delta}\Phi\left(\frac{\|\int_I F(E)d\lambda\|_X}{\mu(E)\lambda(I)}\right)\lambda(I)$$

$$\leqslant \sum_{E\in\pi}\mu(E)\int_S\Phi\left(\frac{\|F(E)\|_X}{\mu(E)}\right)d\lambda$$

by Jensen's inequality, $\leqslant \Sigma_{E\in\pi}\mu(E)\cdot 1 < \infty$. Since $\|F(E)/\mu(E)\|_{L_\Phi(\lambda,X)} \leqslant 1$ and μ is finite. Hence $(f_{\pi,\Delta})$ is an $L_\Phi(\mu\times\lambda, X)$ bounded martingale.

By (i), bounded sets in $L_\Phi(\mu\times\lambda, X)$ are uniformly integrable. Since X has the Radon-Nikodým property, $(f_{\pi,\Delta})$ is $L_1(\mu\times\lambda, X)$ convergent. By the remark at the beginning of the proof, $(f_{\pi,\Delta})$ is $L_\Phi(\mu\times\lambda, X)$ convergent. The remainder of the proof proceeds just as in the L_p case above.

Theorem 2 cannot be generalized even in the case that X is the real line. If Φ fails (ii), Turett [7] has shown that $L_\Phi(\mu)$ contains l_∞ isometrically if μ is nonatomic.

On the other hand, if $\lim_{t\to\infty}\Phi(t)/t$ is finite (this limit exists because Φ is convex), $L_\Phi(\mu)$ is just another renorming of $L_1(\mu)$.

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