ALMOST-COMPLEX SUBSTRUCTURES ON THE SPHERE

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ABSTRACT. The paper solves completely the existence problem of almost-complex substructures on spheres.

1. Introduction. An almost-complex s-substructure on an orientable n-manifold $M$ is defined to be a 2s-dimensional sub-bundle of the tangent-bundle $T(M)$ that admits an almost-complex structure, i.e. a (continuous) structure map $J$, $J^2 = -1$. Assuming a metric on $T(M)$, we call an almost-complex substructure $J$ normalized iff $J$ is also orthogonal on the sub-bundle of its definition. If $J$ is an almost-complex s-substructure, then the orthogonal-component in its polar-decomposition (into an orthogonal and a positive-definite symmetric transformation) defines a normalized almost-complex s-substructure. Analogously, if $w$ is a 2-form on $M$ of constant rank 2s, the union of the 2s-planes on which $w$ is of maximal rank forms a sub-bundle $S_w$ of $T(M)$, and the orthogonal-component of $w$ (regarded as a nonsingular, skew-adjoint transformation on $S_w$) defines a normalized almost-complex s-substructure (e.g. refer to [3]). We thus have: Existence of an almost-complex s-substructure $\iff$ Existence of a normalized almost-complex s-substructure $\iff$ Existence of a 2-form on $M$ of constant rank 2s.

The purpose of this paper is to solve completely the existence problem of almost-complex substructures on spheres. In particular, we shall prove the following Theorem.

Let $c_k$ = dimension of the irreducible (ungraded) representation module for the Clifford-Algebra $C_k$, i.e. $c_k = 2^e_k$ where $e_k =$ number of integers $t$, $0 < t < k$, $t = 0, 1, 2, 4$ (mod 8).

Let $M_k$ be the Atiyah-Todd number defined in [2], i.e. if for an integer $n$ and a prime $p$, $v_p(n)$ denotes the exponent of $p$ in the prime-factorization of $n$, then

$$v_p(M_k) = \begin{cases} \sup(r + v_p(r)), & 1 \leq r \leq \left\lfloor \frac{(k - 1)}{(p - 1)} \right\rfloor \\ 0, & \text{if } p > k. \end{cases}$$

Let $L_k$ be the 2-component of $M_k$, i.e.

$$L_k = 2v_2(M_k)$$

where $v_2(M_k) = \sup(r + v_2(r)), 1 \leq r \leq k - 1$. 

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THEOREM. I. The only almost-complex substructures on even-spheres are the almost-complex structures on $S^2$ and $S^6$.

II. (i) For $s < (2n - 1)/4$; $S^{2n-1}$ admits an almost-complex $s$-substructure iff $c_{2s}/2n$. (ii) For $s > (2n - 1)/4$; $S^{2n-1}$ admits an almost-complex $s$-substructure iff $L_n/L_{n-s}$.

I, II(i) and II(ii) for $n$ odd and $n = 2, 4$ are either known, or else, easy consequences of known material and will be quickly dealt with in §2. The main task of the paper will be to prove II(ii) for $n$ even, $n > 4$.

§3 will reduce the problem to one which can be effectively tackled; and §4 will further reduce it to the order of the obstruction $w_{n,n-s}$ of the complex Stiefel-Fibering.

$W_{n,n-s} \rightarrow W_{n-1,n-s-1} S^{2n-1}$.

The Theorem will then follow from the case $p = 2$ of Theorem I of [4].

2. Elimination of certain special cases. The main tools for this section are the following:

LEMMA 2.1. If $2k < n$, then $S^n$ admits a field of $k$-planes iff it admits a field of $k$-frames.

PROOF. Refer to [5, p. 144, Theorem 27.16].

COROLLARY. Let $S^n$ admit an $s$-substructure.
(i) If $s < n/4$, it admits a field of $2s$-frames. (ii) If $s > n/4$, it admits a field of $(n - 2s)$-frames.

LEMMA 2.2. $S^{2n-1}$ admits an $(n - 1)$-substructure.

PROOF. The canonical complex-structure $i$ on $R^{2n}$ defines a 1-frame, $x \mapsto ix$, $x \in S^{2n-1}$, on $S^{2n-1}$ whose orthogonal-complement in $T(S^{2n-1})$ is preserved under $i$ and thus defines an $(n - 1)$-substructure on $S^{2n-1}$.

We now proceed to prove the easy part of the Theorem.

PROOF OF I. It is well known that $S^2$ and $S^6$ are the only even dimensional spheres that admit almost-complex structures. If $S^{2n}$ admits a proper substructure (i.e. $s < n$), then by the Corollary to Lemma 2.1 it admits at least a 1-frame, which is impossible.

PROOF OF II(i). It is a consequence of the Corollary to Lemma 2.1 and the result of Adams (e.g. refer to [1]) that $S^{2n-1}$ admits a field of $2s$-frames iff $c_{2s}/2n$.

PROOF OF II(ii) FOR $n$ ODD. Suppose $S^{2n-1}$ admits an $s$-substructure for $n$ odd and $s > (2n - 1)/4$. By the Corollary to Lemma 2.1, it admits a field of $(2n - 2s - 1)$-frames; and $2n - 2s - 1 < 1$, i.e. $s > n - 1$ by [1]. Applying Lemma 2.2 yields $s = n - 1$. Since $L_1 = 1$, $L_2 = 2$, this is, precisely, what the Theorem states.

PROOF OF II(ii) FOR $n = 2, 4$. Obvious, since $S^{2n-1}$ is parallelisable.
3. Reduction of the problem.

3.1. Homotopy exact sequence of $SO(2n - 1) \to SO(2n) \to S^{2n-1}$. Consider the exact sequence

$$\pi_{2n-1}SO(2n) \xrightarrow{p_\#} \pi_{2n-1}S^{2n-1} \xrightarrow{\partial} \pi_{2n-2}SO(2n-1) \xrightarrow{\beta_{\#}} \pi_{2n-2}SO \xrightarrow{\varepsilon_{\#}} 0$$

By Theorem 23.4 of [5], $2\iota_{2n-1} \in \text{Im } p_\#$. Hence $\pi_{2n-2}SO(2n - 1) = 0$ or $Z_2$ generated by $\partial (\iota_{2n-1})$. $\pi_{2n-1}SO(2n - 1) = 0$ iff $p$ admits a cross-section iff $S^{2n-1}$ is parallelisable. Thus, for $n$ even, $n > 4$,

$$\pi_{2n-2}SO(2n - 1) = Z_2.$$

As remarked in the introduction, $S^{2n-1}$ admits an $s$-substructure iff it admits a normalized $s$-substructure. A normalized $s$-substructure on $S^{2n-1}$ is, simply, a reduction of the structure-group of $T(S^{2n-1})$ from $SO(2n - 1)$ to $U(s) \times SO(2n - 2s - 1)$; and is, thus, a cross-section to the fibration:

(i) $SO(2n)/U(s) \times SO(2n - 2s - 1) \to S^{2n-1}$.

Define $\Gamma_{m,s} = SO(m)/U(s)$ for $2s \leq m$. We have a fibration:

(ii) $\Gamma_{2n,s} = SO(2n)/U(s) \to \Gamma_{2n-1,s} = SO(2n-1)/U(s) \to S^{2n-1}$.

Let $V_{n,k} = SO(n)/SO(n - k)$ be the Stiefel-Manifold of orthonormal $k$-frames in $R^n$. We have a third fibration:

(iii) $V_{2n,2s+1} \to S^{2n-1}$.

The following proposition will reduce the problem to one which can be effectively tackled.

**Proposition 3.2.** For $n$ even, $n > 4$, $s > (2n - 1)/4$, there exists an $s$-substructure on $S^{2n-1}$ iff the fibration (ii) admits a cross-section.

**Proof.** Let $j: U(s) \subset SO(2n - 1)$ and $k: SO(2n - 2s - 1) \subset SO(2n - 1)$ be the inclusions. Let

$$\pi_{2n-2}U(s) \oplus \pi_{2n-2}SO(2n - 2s - 1) \xrightarrow{(j_{\#}, k_{\#})} \pi_{2n-2}SO(2n - 1) \xrightarrow{\gamma_{\#}} \pi_{2n-2}SO/\gamma_\# \xrightarrow{\beta_{\#}} \pi_{2n-2}SO \xrightarrow{\varepsilon_{\#}} 0$$

be the exact homotopy sequences of the projections:
\( \gamma: SO(2n - 1) \to SO(2n - 1)/U(s) \times SO(2n - 2s - 1), \)

\( \beta: SO(2n - 1) \to \Gamma_{2n-1,s} \) and \( \nu: SO(2n - 1) \to V_{2n-1,2s}. \)

Let \( a_n \in \pi_{2n-2}SO(2n - 1) = \mathbb{Z}_2 \) be the generator and also the obstruction to cross-sectioning

\[ p: SO(2n) \to S^{2n-1}. \]

Then the obstructions to cross-sectioning (i), (ii), and (iii) are \( \gamma_{\#}(a_n) \), \( \beta_{\#}(a_n) \), and \( \nu_{\#}(a_n) \), respectively. The work of Adams [1] shows that \( S^{2n-1} \) does not admit a field of \( 2s \)-frames for \( n > 4 \) and \( s > (2n - 1)/4 \). Thus, (iii) cannot be sectioned and hence \( \nu_{\#}(a_n) \neq 0 \). Since \( \pi_{2n-2}SO(2n - 1) = \mathbb{Z}_2 \), \( \nu \) is a monomorphism. Thus, \( k_{\#} = 0 \) by exactness. Hence (i) admits a cross-section iff \( \gamma_{\#}(a_n) = 0 \), iff \( a_n \in \text{Image}(j_{\#}, k_{\#}) \), iff \( a_n \in \text{Image} j_{\#} \), iff \( \beta_{\#}(a_n) = 0 \), iff (ii) admits a cross-section. Q.E.D.

Cross-sections to (ii) are normalized \( s \)-substructures on \( S^{2n-1} \) whose sub-bundles of definition have “trivial” orthogonal-complements in \( T(S^{2n-1}) \). Such \( s \)-substructures will be called complement-split. We now slightly generalize Proposition 3.2.

**Proposition 3.3.** For \( s > (2n - 1)/4 \), \( S^{2n-1} \) admits an \( s \)-substructure iff it admits a complement-split \( s \)-substructure.

**Proof.** For \( n \) odd, the only possible \( s > (2n - 1)/4 \) such that \( S^{2n-1} \) admits an \( s \)-substructure is \( s = n - 1 \); and a complement-split \( (n - 1) \)-substructure is constructed on \( S^{2n-1} \) in Lemma 2.2. The proof for \( n \) even, \( n > 4 \) is furnished by Proposition 3.2, and is obvious for \( n = 2, 4 \). Q.E.D.

**4. The boundary-operator.** Since \( U(n) \cdot SO(2n - 1) = SO(2n) \) and \( U(n) \cap SO(2n - 1) = U(n - 1) \),

\[ \Gamma_n = SO(2n)/U(n) = SO(2n - 1)/U(n - 1). \]

Let

\[ u: SO(2n - 1) \overset{U(n-1)}{\to} \Gamma_n \to \pi_{2n-1}SO(2n - 1) \overset{u_{\#}}{\to} \pi_{2n-2}\Gamma_n \]

be the exact homotopy sequence of \( u \). We have two more fibrations:

(iv) \[ W_{n,n-s} = U(n)/U(s) \overset{W_{n-1,n-s-1}=U(n-1)/U(s)}{\to} S^{2n-1}. \]

(v) \[ \Gamma_{2n-1,s} = SO(2n - 1)/U(s) \overset{W_{n-1,n-s-1}=U(n-1)/U(s)}{\to} \Gamma_n \]

Let \( w_{n,n-s} \in \pi_{2n-2}W_{n-1,n-s-1} \) denote the obstruction to cross-sectioning fibration (iv). The purpose of this section is to determine the image of \( \partial_{\#} \), and hence further reduce the problem to an equivalent statement about the order of \( w_{n,n-s} \).
Let $H, K$ be closed subgroups of a compact topological group $G$ with $HK = KH$, so that $HK$ is a compact subgroup of $G$. Let

$$
p^1: HK/H \cap K^{K/H \cap K} \to HK/K = H/H \cap K,$$

$$p^2: HK/H \cap K^{H/H \cap K} \to HK/H = KH/H = K/H \cap K$$

be the projections. Let $\partial, \partial^H, \partial^K$ be the boundary-operators of the fibrations:

$$t: HK^{H \cap K} \to HK/H \cap K, \quad t^1: H^{H \cap K} \to H/H \cap K, \quad t^2: K^{H \cap K} \to K/H \cap K$$

respectively. We state the following

**Proposition 4.1.** (i) $\pi q HK/H \cap K = \pi q H/H \cap K \oplus \pi q K/H \cap K$ where $p^1_\#$ and $p^2_\#$ define the projections onto the factors. (ii) Relative to this decomposition, $\partial$ decomposes as a sum of operators:

$$\partial = \partial^H \oplus \partial^K.$$

**Proof.** Elementary exercise.

Let $q^1: HK \to^K H/H \cap K, q^2: HK \to^H K/H \cap K$ denote the projections, i.e. $q^1 = p^1 t$ and $q^2 = p^2 t$.

**Corollary 4.1.** The diagram below is ANTI-commutative.

$$\pi_q HK \quad \xrightarrow{q^1_\#} \quad \pi_q H/H \cap K$$

$$\downarrow q^2_\# \quad \quad \quad \downarrow \partial^H$$

$$\pi_q K/H \cap K \quad \xrightarrow{q^K_\#} \quad \pi_{q-1} H \cap K$$

**Proof.** Let $x \in \pi_q HK$. By Proposition 4.1 $0 = \partial t_\#(x) = \partial^H p^1_\# t_\#(x) + \partial^K p^2_\# t_\#(x) = \partial^H q^1_\#(x) + \partial^K q^2_\#(x)$ i.e. $\partial^H q^1_\# = -\partial^K q^2_\#$. Q.E.D.

**Proposition 4.2.** Let $u: SO(2n - 1) \to^{U(n-1)} \Gamma_n$ and $\partial_u: \pi_{2n-1} \Gamma_n \to \pi_{2n-2} \Gamma(n-1)$ be the boundary-homomorphism in the homotopy exact sequence of $u$. Then $\text{Image } \partial_u$ is the subgroup generated by $2w_{n,n}$, where $w_{n,n}$ is the obstruction to a cross-section of $U(n) \to^{U(n-1)} S^{2n-1}$.

**Proof.** Consider the following diagram.

$$\pi_{2n-1} SO(2n) \quad \xrightarrow{r^\#} \quad \pi_{2n-1} \Gamma_n \quad \xrightarrow{\partial_u} \quad \pi_{2n-2} U(n) = 0$$

$$\downarrow p^\# \quad \quad \quad \downarrow \partial_u$$

$$\pi_{2n-1} S^{2n-1} \quad \xrightarrow{\partial_q} \quad \pi_{2n-2} U(n-1)$$

The row is part of the homotopy exact sequence of $r: SO(2n) \to^{U(n)} \Gamma_n$. The square is anticommutative, and is obtained from Corollary 4.1 by taking

$$H = SO(2n - 1), \quad K = U(n),$$

$$G = HK = KH = SO(2n), \quad H \cap K = U(n - 1).$$

By exactness, $r^\#$ is an epimorphism. Hence

$$\text{Im } \partial_u = \text{Im}(\partial_u r^\#) = \text{Im}(\partial_q p^\#);$$
now use 3.1 and the fact that \( w_{n,n} = \partial_q t_{2n-1} \). Q.E.D.

**Proposition 4.3.** For \( n \) even, \( n > 4 \), the fibration (ii) admits a cross-section iff \( w_{n,n-s} \) is of odd order.

**Proof.** Consider the following commutative diagram, where the bottom row is part of the homotopy exact sequence of the fibration (v).

\[
\begin{array}{cccccc}
\pi_{2n-1} S^{2n-1} & \xrightarrow{\partial_4} & \pi_{2n-2} U(n-1) & \xrightarrow{\partial_5} & \pi_{2n-1} \Gamma_n \\
\downarrow \partial_2 & & \downarrow j_{\#} & & \downarrow b_{\#} \\
\pi_{2n-2} \Gamma_{2n-1,s} & \xrightarrow{\partial_4} & \pi_{2n-2} W_{n-1,n-s-1} & \xrightarrow{\partial_5} & \pi_{2n-1} \Gamma_n \\
\end{array}
\]

(ii) admits a cross-section iff \( \partial_2 = 0 \) iff \( j_{\#} \partial_4 = 0 \) iff \( w_{n,n-s} \in \text{Im} \partial_5 \) and \( \text{Im} \partial_5 = b_{\#}(\text{Im} \partial_4) \). By Proposition 4.2 above and the fact that \( b_{\#}(w_{n,n}) = w_{n,n-s} \), \( b_{\#}(\text{Im} \partial_4) \) is the subgroup generated by \( 2w_{n,n-s} \). Hence \( w_{n,n-s} \in \text{Im} \partial_5 \) iff it is of odd order. Q.E.D.

Finally, let \( p \) be a prime. Theorem I of [4] states that the order of the obstruction \( w_{n,k} \) is prime to \( p \) iff \( n \) is divisible by the \( p \)-primary-component of \( M_k \).

Thus, the proof of II(ii) for \( n \) even, \( n > 4 \), of the Main Theorem follows from Propositions 3.2 and 4.3 and the case \( p = 2 \) of Theorem I of [4].

The proof of the Main Theorem for \( s > (2n - 1)/4 \) is "existential", and, unfortunately, does not give explicit constructions, except for the \((n - 1)\)-substructure on \( S^{2n-1} \) defined in Lemma 2.2, using the complex-structure on \( R^{2n} \). Analogously, for \( n \) even, one can construct an \((n - 2)\)-substructure on \( S^{2n-1} \) using the quaternionic structure on \( R^{2n} \), and these are the only constructions known to the author. It is hoped that one will explicitly construct \( s \)-substructures on \( S^{2n-1} \) for \( s > (2n - 1)/4 \) whenever possible, i.e. when \( L_{n-s}/n \); and this will throw more light onto the problem.

**References**


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