ON CERTAIN EXTREMAL PROBLEMS FOR FUNCTIONS WITH POSITIVE REAL PART

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Abstract. For the class $\mathcal{P}$ of analytic functions $p(z), p(0) = 1$, with positive real part in $|z| < 1$, a type of extremal problems is determined which can be solved already within the set $p(z) = (1 + ez)/(1 - ez), |e| = 1$. One problem of this kind is to find the largest number $\rho(s, \mu)$ such that

$$\Re\left\{ p(z) + szp'(z)/(p(z) + \mu) \right\} > 0,$$

$|z| < \rho(s, \mu)$, for all $p \in \mathcal{P}$, $-1 \neq \mu \in \mathbb{C}, s > 0$. Sharp upper bounds for two other functionals over $\mathcal{P}$ are also given.

1. Introduction. Let $\mathcal{P}$ be the class of analytic functions $p(z)$ in the unit disc $\Delta = \{z | |z| < 1\}$ with $p(0) = 1$, $\Re p(z) > 0$ in $\Delta$. Robertson [2], [3] and Sakaguchi [6] obtained variational formulae for $\mathcal{P}$ and showed that, for fixed $z \in \Delta$, the extreme values of

$$F(p(z), zp'(z)), \quad p \in \mathcal{P},$$

where $F(u, v)$ is analytic in $(u, v) \in \mathbb{C}^2$, $\Re u > 0$, are already attained by the functions

$$\lambda\left(\frac{1 + ez}{1 - ez}\right) + (1 - \lambda)\left(\frac{1 + \bar{e}z}{1 - \bar{e}z}\right), \quad \lambda \in [0, 1], |e| = 1.$$

Zmorović [10], [11] and others used this result to solve extremal problems in $\mathcal{P}$. In many cases of interest, however, even this reduced problem is difficult to solve or appears insurmountable.

In the first part of this paper we present a theorem which characterises a class of extremal problems in $\mathcal{P}$ which can be solved in

$$\mathcal{P}_0 := \{ p \in \mathcal{P} | p(z) = (1 + ez)/(1 - ez), |e| = 1 \}.$$

Our theorem rests on recent results of Suffridge [8] concerning the Hadamard product of starlike functions of order $\alpha$ (compare also Ruscheweyh and Sheil-Small [5] and Ruscheweyh [4]).

One problem which can be treated by our method is to find the largest number $\rho(s, \mu)$ such that for all $p \in \mathcal{P}$

$$\Re\left\{ p(z) + szp'(z)/(p(z) + \mu) \right\} > 0, \quad |z| < \rho(s, \mu),$$

holds, where $-1 \neq \mu \in \mathbb{C}, s > 0$. We shall use this result to solve a problem posed by Yoshikawa [9] and dealing with the ‘radius of spiral-convexity’ of spirallike functions in $\Delta$.

In the last section we shall derive the exact upper bounds of the functionals...
(Re \( p(z) \)/|p(z) + i tan \( \alpha \) and |zp'(z)/(p(z) + i tan \( \alpha \)) for \( p \in P \). Crude bounds, in particular of the second entity, have been used by several authors [1], [3], [7], but a sharp result has not so far been obtained.

2. Let

\[
\begin{align*}
  f(z) &= \sum_{k=0}^{\infty} a_k z^k, \\
  g(z) &= \sum_{k=0}^{\infty} b_k z^k
\end{align*}
\]

be analytic in \( \Delta \). The function

\[
(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k
\]

is also analytic in \( \Delta \) and is called the Hadamard product of \( f \) and \( g \).

**Theorem 1.** For \( p \in P \) let

\[
\hat{p}(z) := \exp\left(\int_0^z \frac{p(\xi)}{\xi} \, d\xi\right).
\]

Let \( F(z) \) be analytic in \( \Delta \), \( F(0) \neq 0 \), and let \( t > 0 \). Then we have for fixed \( z_0 \in \Delta \):

\[
\min_{p \in P} \Re \frac{(F * (p\hat{p}^t))(z_0)}{(F * p)(z_0)} = \min_{p \in P_0} \Re \frac{(F * (p\hat{p}^t))(z_0)}{(F * p^t)(z_0)},
\]

whenever the right-hand side of (3) is nonnegative.

It should be pointed out that

\[
t \frac{(F * (p\hat{p}^t))(z)}{(F * p^t)(z)} = \frac{zg'(z)}{g(z)} + t - 1
\]

where \( g(z) = z(F * p^t)(z) \).

Theorem 1 is almost trivial for \( t = 0 \). The case \( t = 1 \), however, contains the truth of the Pólya-Schoenberg conjecture for convex univalent functions in \( \Delta \).

**Remark 1.** It is easy to see that the right-hand side of (3) is nonnegative if and only if

\[
F * (1 - \epsilon z)/(1 - z)^{1+2t} \neq 0, \quad |z| < |z_0|, \quad |\epsilon| < 1.
\]

This relation is useful for applications of Theorem 1.

Let \( \alpha < 1 \). By \( S^*_\alpha \) we denote the class of functions starlike of order \( \alpha \) in \( \Delta \), i.e. \( f \in S^*_\alpha \) if and only if \( f \) is analytic in \( \Delta \), \( f(0) = f'(0) - 1 = 0 \), and fulfills \( \Re( zf'(z)/f(z)) > \alpha \), \( z \in \Delta \). In particular we have

\[
s_\alpha := z/(1 - z)^{2 - 2\alpha} \in S^*_\alpha.
\]

By \( s_\alpha^{-1} \) we denote that well-defined analytic function which satisfies the relations

\[
s_\alpha^{-1} * s_\alpha = z/(1 - z), \quad s_\alpha^{-1}(0) = 0.
\]

We shall require the following lemmata.
Lemma 1 (Suffridge [7, Theorem 7]). For $\beta < 1$ we have $f \in S_\beta^*$ if and only if $s_\beta^{-1} * f * g \in S_\beta^*$ for all $g \in S_\beta^*$.

Lemma 2 (Suffridge [7, Theorem 10]). Let $\alpha \leq \beta < 1$. Then $f \in S_\alpha^*$ implies $s_\alpha^{-1} * s_\beta * f \in S_\beta^*$.

For $\alpha \leq \beta < 1$ we introduce the set $V_{\alpha \beta}$ of analytic functions $h(z)$ in $\Delta$, $h(0) = 0$, such that $h * f \in S_\beta^*$ for each $f \in S_\alpha^*$.

Lemma 3. Let $h(z)$ be analytic in $\Delta$, $h(0) = 0$. Then $h \in V_{\alpha \beta}$ if and only if $h * s_\alpha \in S_\beta^*$.

Proof. Let $g \in S_\beta^*$, $h := s_\alpha^{-1} * g$. Then for each $f \in S_\alpha^*$,

$$f * h = (s_\alpha^{-1} * s_\beta * f) * s_\beta^{-1} * g \in S_\beta^*$$

by Lemma 1, since $s_\alpha^{-1} * s_\beta * f \in S_\beta^*$ by Lemma 2. On the other hand, $h \in V_{\alpha \beta}$ implies $h * s_\alpha \in S_\beta^*$ since $s_\alpha \in S_\alpha^*$.

Proof of Theorem 1. Lemma 3 shows that for $\alpha \leq \beta < 1$ and a certain $h(z)$ analytic in $\Delta$, $h(0) = 0$, we have

$$\text{Re} \frac{z(h * g)'(z)}{(h * g)(z)} > \beta \quad \text{for all } g \in S_\alpha^*, z \in \Delta,$$

if and only if

$$\text{Re} \frac{z(h * s_\alpha)'(z)}{(h * s_\alpha)(z)} > \beta, \quad z \in \Delta.$$

Now let $p \in P$, $t > 0$. Obviously $z\hat{p}'(z) \in S_{1-t}^*$, and, in particular, for $p_0(z) = (1 + z)/(1 - z)$ we have $z\hat{p}'_0(z) = s_{1-t}(z)$. In this notation, (5) and (6) read $(1 - t < \beta < 1)$:

$$\text{Re} t \frac{(h * (z\hat{p}'))(z)}{(h * z\hat{p}')(z)} > \beta + t - 1 \quad \text{for all } p \in P, z \in \Delta,$$

if and only if

$$\text{Re} t \frac{(h * (z p_0\hat{p}'))(z)}{(h * z\hat{p}')(z)} > \beta + t - 1, \quad z \in \Delta.$$

Now assume

$$\min_{p \in P_0} \text{Re} \frac{(F * p\hat{p}')(z_0)}{(F * \hat{p}')(z_0)} = v \geq 0$$

and put $h(z) := zF(z_0z)$, $v = (\beta + t - 1)/t$. Thus (9) implies (8) and, hence, (7), which in turn may be written as

$$\min_{p \in P} \text{Re} \frac{(F * p\hat{p}')(z_0)}{(F * \hat{p}')(z_0)} = v.$$

The case $t = 0$ is well known and follows from the Herglotz representation of the functions in $P$. 

3. In this section we give a simple application of Theorem 1. For \(-1 \neq \mu \in \mathbb{C}, s > 0\), let

\[
F(z) := \left( \frac{\mu + 1}{s} - 1 \right) \frac{1}{1 - z} + \frac{1}{(1 - z)^2}.
\]

Then for \(p \in P\) we obtain

\[
\frac{(F \ast (p^s))'(z)}{(F \ast \hat{p}^s)(z)} = p(z) + s \frac{zp'(z)}{p(z) + \mu}.
\]

**Corollary 1.** For \(p \in P\) and

\[
|z| < \frac{|\mu + 1|}{A + \left( A^2 - |\mu^2 - 1|^2 \right)^{1/2}}, \quad A = 2(s + 1)^2 + |\mu|^2 - 1,
\]

we have

\[
\text{Re} \left( p(z) + s \frac{zp'(z)}{p(z) + \mu} \right) > 0.
\]

The bound given by (12) is best possible.

**Proof.** From Remark 1 we deduce that we only have to find the largest radius \(\rho = \rho(s, \mu)\) such that for \(F\), given by (11),

\[
F \ast \frac{1 - \varepsilon z}{(1 - z)^{1+2/s}} \neq 0, \quad |z| < \rho(s, \mu), |\varepsilon| < 1.
\]

This is equivalent to the condition

\[-s \frac{\varepsilon z}{1 - \varepsilon z} + (s + 2) \frac{z}{1 - z} \neq -\mu - 1, \quad |z| < \rho(s, \mu), |\varepsilon| < 1,
\]

and thus with \(|2\rho^2 + (\mu + 1)(1 - \rho^2)| > 2\rho(s + 1)\). This leads immediately to (12).

It should be mentioned that one can use Theorem 1 to compute

\[
m(r) = \min_{p \in P} \min_{|z| < r} \text{Re} \left( p(z) + s \frac{zp'(z)}{p(z) + \mu} \right)
\]

as long as \(m(r)\) is nonnegative.

Let \(f(z)\) be analytic in \(\Delta, f(0) = f'(0) - 1 = 0\). Yoshikawa [9] called \(f(z)\) "\(\gamma\)-spiral-convex" if it satisfies the condition

\[
\text{Re} \left( 1 + \frac{zf''(x)}{f'(x)} + i \tan \frac{\gamma f'(x)}{f'(x)} \right) > 0
\]

in \(\Delta\) and gave a geometrical interpretation of this condition. He proved that every \(\gamma\)-spiral-convex function is \(\gamma\)-spirallike, i.e. it fulfils

\[
\text{Re} \ e^{i\gamma} \frac{z f'(z)}{f(z)} > 0, \quad z \in \Delta.
\]

He posed the problem to find the largest radius \(\rho = \rho(\gamma)\), such that every \(\gamma\)-spirallike function is \(\gamma\)-spiral-convex in \(|z| < \rho(\gamma)\), thereby generalizing the
radius of (0-spiral-)convexity of the starlike (= 0-spirallike) functions in $\Delta$.

It is easily seen that this problem is equivalent to the following question: What is the greatest number $\rho(\gamma)$ such that

$$\text{Re} \left( \frac{zp'(z)}{p(z) + i \tan \gamma} \right) > 0, \quad |z| < \rho(\gamma),$$

for all $p \in P$. This question has been answered in Corollary 1 and thus we can state

**Theorem 2.** The greatest common radius $\rho(\gamma)$ of $\gamma$-spiral-convexity of $\gamma$-spirallike functions in $\Delta$ is given by

$$\rho(\gamma) = \sqrt{1 + 3 \cos^2 \gamma} - \sqrt{3} \cos \gamma. \tag{16}$$

Yoshikawa himself was able to show that the value given in (16) is an upper bound for $\rho(\gamma)$.

4. In this last section we derive upper bounds of two special functionals on $P$. The method used is completely apart from those discussed in the previous chapters. The extremal functions, however, belong to $P_0$ too.

**Theorem 3.** For $p \in P$ and $\alpha \in (-\pi/2, \pi/2)$ the estimate

$$\frac{\text{Re} p(z)}{|p(z) + i \tan \alpha|} \leq \begin{cases} \frac{(1 - |z|^2)\cos \alpha}{1 - 2|z| \sin \alpha + |z|^2}, & |z| < \tan \frac{\alpha}{2}, \\ 1, & |z| > \tan \frac{\alpha}{2} \end{cases},$$

is valid and sharp. Equality holds for certain functions in $P_0$.

The estimate $|p'(z)| \leq (2 \text{ Re} p(z))/(1 - |z|^2)$ holds for every $p \in P$ and $z \in \Delta$ (see [3]). Equality holds only for every $p \in P_0$ and $z \in \Delta$. Thus Theorem 3 has the following corollary.

**Corollary 2.** For $p \in P$ and $\alpha \in (-\pi/2, \pi/2)$ the estimate

$$\left| \frac{zp'(z)}{p(z) + i \tan \alpha} \right| \leq \begin{cases} \frac{2|z| \cos \alpha}{1 - 2|z| \sin \alpha + |z|^2}, & |z| < \tan \frac{\alpha}{2}, \\ \frac{2|z|}{1 - |z|^2}, & |z| > \tan \frac{\alpha}{2} \end{cases},$$

is valid and sharp. Equality holds only for certain functions $p \in P_0$.

We shall require the following well-known fact: The range of the functional $w = p(z)$, $z$ fixed, over $P$ is the closed disc $K_{|z|}$ with center $(1 + |z|^2)/(1 - |z|^2)$ and radius $2|z|/(1 - |z|^2)$. Furthermore, the points on $\partial K_{|z|}$ correspond only to functions in $P_0$. The estimate

$$\frac{\text{Re} p(z)}{|p(z) + i \tan \alpha|} \leq \frac{\text{Re} p(z)}{\text{Re} p(z)} = 1,$$

and equality can hold if and only if there exists a $w_1 \in \partial K_{|z|}$ with $\text{Im} w_1 = -\tan \alpha$. This is the case if and only if $|z| \geq |\tan (\alpha/2)|$ and equality holds for a certain $p \in P_0$. This proves the bottom half of (17).

Now let $z_0 \in \Delta$ be fixed, $|z_0| < |\tan (\alpha/2)|$. Let $p_0 \in P$ be an extremal
function for our functional at $z_0$ ($P$ is a normal family). The function

$$p^*(z) = \frac{p_0(z) + i\lambda}{1 + i\lambda p_0(z)}$$

belongs to $P$ for arbitrary $\lambda \in \mathbb{R}$ and gives the variational formula

$$p^*(z) = p_0(z) + i\lambda(1 - p_0^2(z)) + o(\lambda).$$

Using the extremal property of $p_0$ and this variation we are led to the condition $\cos \alpha |p_0(z_0) + i \tan \alpha| = 1$. If we let $p_0(z_0) + i \tan \alpha = \sec \alpha e^{i\gamma}$ it follows for $p \in P$

$$\frac{\text{Re} p(z_0)}{|p(z_0) + i \tan \alpha|} \leq \frac{\text{Re} p_0(z_0)}{|p_0(z_0) + i \tan \alpha|} = \cos \gamma,$$

and we are left with the problem to maximize $\cos \gamma$ within the set of points $\sec \alpha e^{i\gamma} - i \tan \alpha \in K_{|z_0|}$. It is easily seen that the extremal value is

$$\frac{(1 - |z_0|^2)\cos \alpha}{1 - 2|z_0| |\sin \alpha| + |z_0|^2}$$

and corresponds to a boundary point of $K_{|z_0|}$. The theorem is proven.


**References**


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