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STRICT TOPOLOGY AND P-SPACES

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ABSTRACT. For a completely regular Hausdorff space X and a normed space E, let $C_b(X, E)$ be the space of all bounded continuous functions from X into E with strict topology β_0. It is proved that if X is a P-space, $(C_b(X, E), \beta_0)$ is Mackey; if, in addition, E is complete, then $(C_b(X, E), \beta_0)$ is strongly Mackey.

In this paper, X denotes a completely regular Hausdorff space, K the field of real or complex numbers (we shall call them scalars), $C_b(X)$ all scalar-valued bounded continuous functions on X, $(E, \|\cdot\|)$ a normed space over K, $C_b(X, E)$ all bounded continuous functions from X into E, and E' the topological dual of X. We denote by $\langle\ ,\ \rangle$ the natural bilinear form on $E \times E'$ or $E' \times E$. All vector spaces are taken over K. Let $\mathfrak{B}(X)$ be all Borel subsets of X and $M_t(X)$ all tight scalar-valued Borel measures on X [1], [4], [10]. We put

$$M_t(X, E') = \{\mu\colon \mathfrak{B}(X) \to E'\colon \mu \text{ finitely additive, and } |\mu| \in M_t(X), \text{ where for any } B \in \mathfrak{B}(X), |\mu|(B) = \sup\{\textstyle\sum |\langle \mu(B_i), x_i\rangle|\colon \{B_i\} \text{ a finitely Borel partition of } B \text{ and } \{x_i\} \subset E \text{ with } \|x_i\| \leqslant 1, \forall i\}\}$$

(see [1], [4]). For a $\mu \in M_t(X, E')$ and $x \in E$, $\mu_x\colon \mathfrak{B}(X) \to K$, defined by $\mu_x(B) = \langle \mu(B), x\rangle$, $B \in \mathfrak{B}(X)$, is in $M_t(X)$. Integration with respect to a $\mu \in M_t(X, E')$ is taken in the sense of [1]. For a $\mu \in M_t(X, E')$ and $f \in C_b(X, E)$, $|\mu(f)| \leqslant |\mu|\,(\|f\|)$, where $\|f\|\colon X \to R$, $\|f\|(x) = \|f(x)\|$ [1, p. 851].

The strict topology β_0 on $C_b(X, E)$ is defined by the family of seminorms $\|\cdot\|_h$, as h varies through all scalar-valued functions on X, vanishing at infinity, $\|f\|_h = \sup_{x \in X} \|h(x) f(x)\|$, $f \in C_b(X, E)$. It is proved in [1] that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$, $(C_b(X, E), \beta_0)' = M_t(X, E')$, and β_0 is the finest locally convex topology which coincides with compact-open topology on norm-bounded subsets of $C_b(X, E)$; also bounded subsets of $(C_b(X, E), \beta_0)$

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are norm-bounded. (For $E = K = R$ this result is proved in [9], but it immediately carries over to the case when E is a normed space since $M_t(X, E')$ is a closed subspace of the Banach space $(C_b(X, E), \|\cdot\|)'$.) Considering $M_t(X, E')$ a Banach space, with norm induced by $(C_b(X, E), \|\cdot\|)'$, we have $\|\mu\| = |\mu|(X)$, $\forall \mu \in M_t(X, E')$ (it is a simple verification, cf. [4, p. 315]).

A completely regular Hausdorff space X is called a P-space if every G_δ set is open in X [2, p. 63]. In this paper we prove that if X is a P-space then $(C_b(X, E), \beta_0)$ is Mackey; if, in addition, E is complete, then E is strongly Mackey. A Hausdorff locally convex space G is called strongly Mackey if every $\sigma(G', G)$ relatively countably compact subset of G' is equicontinuous (we refer to [8] for locally convex spaces).

We first prove the following lemmas.

LEMMA 1. *Let 2^N denote all subsets of N, with product topology. If $\lambda_n: 2^N \to K$ is a sequence of countably additive measures (this implies they are continuous) and $\lim \lambda_n(M) = \lambda(M)$ exists $\forall M \subset N$, then $\lambda_n \to \lambda$ uniformly on 2^N.*

PROOF. This is a particular case of [6, Lemma 1]. To prove this we have only to note that by Osgood's theorem [5, p. 86], the sequence $\{\lambda_n\}$ is equicontinuous at some point of 2^N. For completeness we give details.

Since $\{0, 1\}$ is a topological group, with discrete topology ($1 + 1 = 0$, mod 2), $G = 2^N = \{0, 1\}^N$, with product topology, is also a topological group, which we write additively with neutral element 0. Fix $\varepsilon > 0$ and suppose λ_n's are equicontinuous at $p \in G$. There exist a 0-nbd

$$V = \left(\prod_{i=1}^{m} \xi_i\right)\left(\prod_{j=m+1}^{\infty} J_j\right),$$

where $\xi_i = \{0\}$, $1 \leqslant i \leqslant m$, and $J_j = \{0, 1\}$, $m + 1 \leqslant j < \infty$, such that

$$(*) \qquad |\lambda_n(p + V) - \lambda_n(p)| < \varepsilon/8, \quad \forall n.$$

Let $p = (p_1, p_2, \ldots, p_m, p_{m+1}, \ldots)$ and put $p' = (p_1, p_2, \ldots, p_m, 0, 0, \ldots)$ and $p'' = (0, 0, \ldots, 0, p_{m+1}, p_{m+2}, \ldots)$. $p = p' + p''$. Fix $v \in V$ and take $v' \in V$ such that $p'' + v' = v$. From $(*)$ we get $|\lambda_n(p' + p'' + v') - \lambda_n(p' + p'')| < \varepsilon/8$ and so $|\lambda_n(p') + \lambda_n(v) - \lambda_n(p') - \lambda_n(p'')| < \varepsilon/8$ (note λ_n's are additive). This gives $|\lambda_n(v) - \lambda_n(p'')| < \varepsilon/8$, $\forall v \in V$. In particular, $|\lambda_n(p'')| < \varepsilon/8$. Combining these results we get $|\lambda_n(V)| < \varepsilon/4$, $\forall n$. Since $\mathfrak{A}_0 = \{1, 2, \ldots, m\} \cap 2^N$ (i.e., subsets of $\{1, 2, \ldots, m\}$) is finite there exists a positive integer n_0 such that $|\lambda_n(A) - \lambda(A)| < \varepsilon/4$, $\forall n \geqslant n_0$ and $A \in \mathfrak{A}_0$. Take $q \in 2^N$, $q = (q_1, q_2, \ldots, q_m, q_{m+1}, \ldots)$ and put $q' = (q_1, q_2, \ldots, q_m, 0, 0, \ldots)$, $q'' = (0, 0, \ldots, 0, q_{m+1}, q_{m+2}, \ldots)$. Then $q'' + q' = q$ and $q'' \in V$. For $n \geqslant n_0$,

$$\begin{aligned} |\lambda_n(q) - \lambda(q)| &\leqslant |\lambda_n(q') - \lambda(q')| + |\lambda_n(q'') - \lambda(q'')| \\ &\leqslant \varepsilon/4 + |\lambda_n(q'')| + |\lambda(q'')| \leqslant \varepsilon/4 + \varepsilon/4 + \varepsilon/4 < \varepsilon. \end{aligned}$$

This proves the result.

A subset $A \subset M_t(X, E')$ is said to be uniformly tight if given $\varepsilon > 0$, there exists a compact subset $K \subset X$ such that $|\mu|(X \setminus K) \leqslant \varepsilon$, $\forall \mu \in A$.

LEMMA 2. *A subset $A \subset M_t(X, E')$ is β_0-equicontinuous iff A is uniformly tight and norm-bounded.*

PROOF. Let A be norm-bounded and uniformly tight. Put $\alpha_0 = \sup\{\|\mu\|: \mu \in A\}$. Since β_0-topology is the finest locally convex topology, coinciding with compact-open topology on norm-bounded subsets of $C_b(X, E)$, it is enough to prove that for any $k > 0$ there exists a compact subset K of X and some $\eta > 0$ such that

$$Z = \{f \in C_b(X, E): \|f\| \leqslant k, \|f\|_{|K} \leqslant \eta\}$$
$$\subset \{g \in C_b(X, E): |\mu(g)| \leqslant 1, \forall \mu \in A\}.$$

By uniform tightness of A, there exists a compact $K \subset X$ such that $|\mu|(X \setminus K) < 1/(2k+1)$, $\forall \mu \in A$. Take $\eta = 1/2(1+\alpha_0)$. For an $f \in Z$ and $\mu \in A$,

$$|\mu(f)| \leqslant \int \|f\| \, d|\mu| = \int_K \|f\| \, d|\mu| + \int_{X \setminus K} \|f\| \, d|\mu|$$
$$\leqslant \alpha_0/2(1+\alpha_0) + k/(2k+1) \leqslant 1.$$

This proves A is β_0-equicontinuous.

Conversely, if $A \subset M_t(X, E')$ is β_0-equicontinuous then A is norm-bounded, since $\beta_0 \leqslant \|\cdot\|$ on $C_b(X, E)$. Fix $\varepsilon > 0$. There exists a scalar-valued function φ on X such that

$$\{f \in C_b(X, E): \|f\varphi\| \leqslant 1\} \subset \{g \in C_b(X, E): |\mu(g)| \leqslant 1, \forall \mu \in A\}.$$

Take a compact set K, in X, with the property that $K \supset \{x \in X: |\varphi(x)| \geqslant \varepsilon\}$. If $|\mu|(X \setminus K) > \varepsilon$, for some $\mu \in A$, then, by using the fact that $\mu_x \in M_t(X)$, $\forall x \in E$, we get a finite disjoint collection $\{C_i\}$ of compact subsets of $X \setminus K$ and $\{x_i\} \subset E$, with $\|x_i\| \leqslant 1$, $\forall i$, such that $|\sum \langle \mu(C_i), x_i \rangle| > \varepsilon$. This means there is a collection $\{f_i\} \subset C_b(X)$, $0 \leqslant f_i \leqslant 1$, $\forall i$, supports of f_i's mutually disjoint, $f_i = 0$ on K, $\forall i$, such that $|\mu(f)| > \varepsilon$, where $f = \sum f_i \otimes x_i$. Now $\|f\varphi\| \leqslant \varepsilon$ implies $|\mu(f)| \leqslant \varepsilon$, which is a contradiction. This proves the result.

LEMMA 3. *Let A be a norm-bounded, relatively countably compact subset of $(F', \sigma(F', F))$, where $F = C_b(X, E)$ and $F' = M_t(X, E')$, and assume that X is a P-space. Then A is equicontinuous on (F, β_0).*

PROOF. First we note that $\mu \in M_t(X)$ implies $|\mu| \in l^1(X)$, since X is a P-space [12, p. 467]. Given $\varepsilon > 0$, we prove the existence of a finite subset $D \subset X$ such that $|\mu|(X \setminus D) < \varepsilon$, $\forall \mu \in A$. Suppose this is not true. Take a $\mu_1 \in A$ and a finite set $C_1 \subset X$ such that $|\mu_1|(X \setminus C_1) < \varepsilon/2$. We get a $\mu_2 \in A$ such that $|\mu_2|(X \setminus C_1) \geqslant \varepsilon$. Take a finite subset C_2 of X, $C_2 \supset C_1$ such that

$|\mu_2|(X\backslash C_2) < \varepsilon/2$. Continuing this process we get a sequence $\{\mu_n\} \subset A$, and an increasing sequence $\{C_n\}$ of finite subsets of X such that $|\mu_n|(X\backslash C_i) < \varepsilon/2$ for $i \geqslant n$ and $|\mu_n|(X\backslash C_i) \geqslant \varepsilon$ for $1 \leqslant i \leqslant n-1$. Putting $C_0 = \varnothing$ and $D_i = C_i \backslash C_{i-1}$ $(i = 1, 2, \ldots)$, we get

$$|\mu_n|(D_n) = |\mu_n|(C_n \backslash C_{n-1}) = |\mu_n|((X\backslash C_{n-1})\backslash(X\backslash C_n)) \geqslant \varepsilon/2.$$

Since $\{D_n\}$ is a disjoint sequence of finite subsets of X, for every n, there exists a finite partition $\{A_i^{(n)}\colon 1 \leqslant i \leqslant p_n\}$ of D_n, and points $\{x_i^{(n)}\colon 1 \leqslant i \leqslant p_n\}$ in the closed unit ball of E such that

$$|\mu_n|(D_n) < |\mu_n|\left(\sum_{i=1}^{p_n} x_i^{(n)} \otimes \chi_{A_i^{(n)}}\right) + \frac{\varepsilon}{4}.$$

Since X is a P-space and $\{A_i^{(n)}\colon 1 \leqslant i \leqslant p_n\ (1 \leqslant n < \infty)\}$ is a countable collection of disjoint finite subsets of X, $\exists$ a disjoint collection of clopen subsets $\{U_i^{(n)}\colon 1 \leqslant i \leqslant p_n\ (1 \leqslant n < \infty)\}$ of X such that $U_i^{(n)} \supset A_i^{(n)}$ and $\mu_n(x_i^{(n)} \otimes \chi_{A_i^{(n)}}) = \mu_n(x_i^{(n)} \otimes \chi_{U_i^{(n)}})$, $\forall n$, and $\forall i$ (this follows from the regularity of μ_x, $\mu \in M_t(X, E')$, $x \in E$, and the fact that X is a P-space). Putting $f_n = \sum_{i=1}^{p_n} x_i^{(n)} \otimes \chi_{U_i^{(n)}}$, we get $|\mu_n(f_n)| > \varepsilon/4$, $\forall n$ and $f_n \in C_b(X, E)$. For any subset $M \subset N$, $\sum_{n \in M} f_n = f_M \in C_b(X, E)$ and $\|f_M\| \leqslant 1$ (here again we are using the fact that X is a P-space). The space $H = \{f_M\colon M \subset N\}$ with topology induced by $\sigma(F, F')$, contains $\{f_P\colon P \subset N, P \text{ finite}\}$ as a dense subset–to prove this, fix $M \subset N$ and put $g_m = \sum_{i \in \{1,2,\ldots,m\} \cap M} f_i$; this gives $|\mu(f_M - g_m)| \leqslant |\mu|(\|f_M - g_m\|) \to 0$, by the dominated convergence theorem, $\forall \mu \in F'$. Also A, considered as a set of continuous functions on H, with the topology of pointwise convergence, is relatively countably compact, and so by [7] there exists a subsequence of $\{\mu_n\}$, which again we denote by $\{\mu_n\}$, such that $\{\mu_n\}$ is convergent pointwise on H. Define $\lambda_n\colon 2^N \to K$, $\lambda_n(M) = \mu_n(f_M)$. It is easy to verify that λ_n's are countably additive and $\lim \lambda_n(M) = \lambda(M)$ exists $\forall M \subset N$. By Lemma 1, $\lambda_n \to \lambda$ uniformly on 2^N. Choose $n_0 \in N$ so large that $|\lambda(\{n\})| < \varepsilon/10$ and $\forall P \in 2^N$, $|\lambda_n(P) - \lambda(P)| < \varepsilon/10$, $\forall n \geqslant n_0$. In particular, $|\lambda_{n_0}(\{n_0\}) - \lambda(\{n_0\})| < \varepsilon/10$, and so $|\lambda_{n_0}(\{n_0\})| < \varepsilon/5$, i.e., $|\mu_{n_0}(f_{n_0})| < \varepsilon/5$. This contradicts $|\mu_n(f_n)| > \varepsilon/4$, $\forall n$. Using Lemma 2, we get the result.

EXAMPLE 4. The condition that A, in Lemma 3, be norm-bounded is essential. Let E be the subspace of l_1 over reals, consisting of sequences with only finite number of nonzero components with induced norm. In $E' = l_\infty$, for every positive integer n, let y_n have all components 0 except nth which is equal to n. Put $A = \{y_n\}$. Now $y_n \to 0$ in $(E', \sigma(E', E))$, but, being unbounded, is not equicontinuous. Thus E is Mackey but not strongly Mackey. Take $X = \{x_0\}$, a one-point set. Then $(C_b(X, E), \beta_0)$ is isometric isomorphic to E. Thus Lemma 3 cannot hold without the assumption of norm-boundedness on A.

THEOREM 5. *If X is a P-space and E a normed space, then* $(C_b(X, E), \beta_0)$ *is*

Mackey. If, in addition, E is complete (i.e., E is a Banach space) then $(C_b(X, E), \beta_0)$ is strongly Mackey.

PROOF. Let A be an absolutely convex, compact subset of $(F', \sigma(F', F))$, where $F = (C_b(X, E), \beta_0)$, $F' = M_t(X, E')$. Since the bounded subsets of $(C_b(X, E), \beta_0)$ are norm-bounded, the strong topology on $M_t(X, E')$ is the norm topology and so A is norm-bounded [8, 5.1, p. 141]. By Lemma 3, A is equicontinuous. If E is a Banach space, then $G = (C_b(X, E), \|\cdot\|)$ is also a Banach space and $M_t(X, E') \subset G'$. Thus if A is a relatively countably compact subset of $(M_t(X, E'), \sigma(M_t(X, E')), C_b(X, E))$, then A is a relatively countably compact subset of $(G', \sigma(G', G))$ and so is norm-bounded. Lemma 3 now gives the result. This completes the proof.

REMARK 6. Our proof is different from the usual proof that the function space be Mackey; the usual proof starts out with "gliding hump" argument and then uses l_∞ trick [11]. This theorem generalizes the main result of [11].

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