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A NONINVERTIBLE UNWEIGHTED BILATERAL SHIFT

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ABSTRACT. A noninvertible unweighted bilateral shift is constructed in a Nakano l^{p_i} space.

To Hilbert space operator theorists our title may seem paradoxical. To them there is only one (up to unitary equivalence) "unweighted bilateral shift", the operator S defined by

$$S \sum_{-\infty}^{\infty} b_n y_n = \sum_{-\infty}^{\infty} b_n y_{n+1} \tag{1}$$

where $\{y_n\}_{-\infty}^{\infty}$ is an orthonormal basis in Hilbert space and the b_n are scalars. That operator is unitary. To get the example of our title, (1) will instead define an operator on Banach space while $\{y_n\}_{-\infty}^{\infty}$ will be a Schauder basis with $\|y_n\| = 1$ for all n (the unweightedness).

Given a space B with Schauder basis $\{y_n\}_{-\infty}^{\infty}$, let us interpret (1) as defining a linear operator S with domain precisely that linear manifold of B on which the formula makes sense. The operator S is defined by a matrix with respect to a Schauder basis and hence [4, 11.3, Corollary 5] is closed. Also, it is clear that the domain of S is dense. Hence the closed graph theorem may be applied to yield the following

CRITERION. S is bounded iff for each element $\Sigma_{-\infty}^{\infty} b_n y_n$ in B, $\Sigma_{-\infty}^{\infty} b_n y_{n+1}$ also is an element of B. Equivalently, S is unbounded iff there is an element $\Sigma_{-\infty}^{\infty} b_n y_n$ of B such that $\Sigma_{-\infty}^{\infty} b_n y_{n+1}$ does not converge to an element of B.

S is 1-1 and hence has a left inverse S^{-1} defined by

$$S^{-1} \sum_{-\infty}^{\infty} b_n y_n = \sum_{-\infty}^{\infty} b_n y_{n-1}.$$

An analogous criterion can determine the boundedness of S^{-1}.

The space of our example was invented by Nakano.

THEOREM. (*See* [2] *and* [3, §89] *for the proofs of the nontrivial parts of this theorem.*) *Given a bounded sequence* $\{p_n\}_{-\infty}^{\infty}$ *of real numbers* $\geqslant 1$, *the collection* B *of complex sequences* $\{b_n\}_{-\infty}^{\infty}$, *such that* $\Sigma_{-\infty}^{\infty} |b_n|^{p_n} < \infty$, *forms a Banach space under the usual algebraic operations and the norm* $\|\{b_n\}_{-\infty}^{\infty}\| = N$, *where* N *is the unique nonnegative real such that* $\Sigma_{-\infty}^{\infty} |b_n/N|^{p_n} = 1$. *If* y_n *is the sequence with* 1 *in the nth place and* 0 *in the other places then* $\{y_n\}_{-\infty}^{\infty}$ *is a Schauder basis in* B. $\{y_n\}_{-\infty}^{\infty}$ *is an absolute basis (i.e. if* $\Sigma b_n y_n \in B$ *and*

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$|c_n| \leqslant M$, *all* n, *then* $\sum c_n b_n y_n \in B$).

EXAMPLE. Let B be defined as in the above theorem with

$$p_n = 1 \quad \text{for } n \leqslant 1,$$

$$p_n = 2((2^k - 1)/n) \quad \text{for } n \geqslant 2, \text{ where } 2^k - 1 \leqslant n \leqslant 2^{k+1} - 2.$$

Then (a) S is bounded but (b) S^{-1} is unbounded.

PROOF. Let $\alpha_n = p_n/p_{n-1}$. For $n \leqslant 2$, $\alpha_n = 1$. For $n > 2$, $\alpha_n = (n-1)/n = 1 - (1/n)$ unless $n = 2^k - 1$ in which case $p_n = 2, p_{n-1} = 1$ and $\alpha_n = 2$.

(a) Apply the criterion. Suppose $\{b_n\}_{-\infty}^{\infty} \in B$. That is $\sum v_n < \infty$ where $v_{n+1} = |b_n|^{p_n}$. It must be shown that $\sum |b_{n-1}|^{p_n} < \infty$. But

$$\sum |b_{n-1}|^{p_n} = \sum \left(|b_{n-1}|^{p_{n-1}}\right)^{(p_n/p_{n-1})} = \sum v_n^{\alpha_n}$$

$$\leqslant \sum_{-\infty}^{2} v_n + \sum_{3}^{\infty} v_n^2 + \sum_{3}^{\infty} v_n^{1-(1/n)}.$$

The convergence of $\sum v_n$ implies the convergence of each of the last three sums; only the third sum needs checking. If $v_n \leqslant e^{-n}$ then $v_n^{1-(1/n)} \leqslant e^{-n(1-(1/n))} = e^{-n+1}$. If $v_n \geqslant e^{-n}$ then $v_n^{1/n} \geqslant e^{-1}$, so $v_n^{1-(1/n)} \leqslant e v_n$. Thus the third sum converges too.

(b) We must find a sequence $\{b_n\}_{-\infty}^{\infty}$ such that $\sum |b_n|^{p_n} < \infty$ and $\sum |b_n|^{p_{n-1}} = \infty$. But $\sum |b_n|^{p_{n-1}} = \sum (|b_n|^{p_n})^{\alpha_n^{-1}}$ and since $\alpha_n^{-1} = \frac{1}{2}$ for an infinite number of n there is no difficulty in finding such a sequence.

COMMENT. It was conjectured in [1] that the weighted shift in Banach space

$$S_{\mathbf{a}} \sum_{-\infty}^{\infty} b_n y_n = \sum_{-\infty}^{\infty} a_{n+1} b_n y_{n+1} \tag{2}$$

(for a fixed sequence of never-zero weights $\mathbf{a} = \{a_n\}_{-\infty}^{\infty}$) has its spectrum determined by the sequence $\mathbf{a}$ alone, regardless of the space or the basis. The precise conjecture was that if $S_{\mathbf{a}}$ is bounded, spectrum $S_{\mathbf{a}}$ is the closed zero-centered annulus with radii

$$\lim_{n\to\infty} \left(\sup_m \prod_{i=m+1}^{m+n} |a_i| \right)^{1/n} \quad \text{and} \quad \lim_{n\to\infty} \left(\inf_m \prod_{i=m+1}^{m+n} |a_i| \right)^{1/n}.$$

According to this formula, spectrum S should be the unit circle. However, it can be shown that spectrum S for the above example is a zero-centered disc. Can an example be constructed for which spectrum S is not symmetric about the origin?

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