ON THE FACIAL STRUCTURE OF A CONVEX BODY

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Abstract. The family formed by taking the relative interior of each face of a $d$-dimensional convex body $C$ is a partition of $C$. It is shown here that the subfamily consisting of all the $(d - 2)$-dimensional sets has a quotient topology which is paracompact and this is used to prove a property of the set of extreme points when $d = 3$.

A convex body in $E^d$ is any closed bounded convex subset $C$ with nonempty interior. A flat is any translate of a subspace. A face of $C$ is a convex subset $F$ which is the intersection of $C$ with a flat such that $C \setminus F$ is also convex. Thus the extreme points of $C$ are exactly the 0-dimensional faces of $C$. The set of extreme points of $C$ will be denoted by $\text{ext } C$ and for any set $A$ the boundary of $A$ and the closure of $A$ will be denoted by $\text{bd } A$ and $\text{cl } A$, respectively.

It is easily seen that if $C$ is 2-dimensional, then $\text{ext } C$ is a closed set. This need not be true in higher dimensions. In fact, there is a convex body $C$ in $E^3$ for which $\text{ext } C$ and $\text{bd } C \setminus \text{ext } C$ are both dense in $\text{bd } C$ (see [4, p. 104]). However, we can show the following relationship between $\text{ext } C$ and its closure:

**Theorem 1.** If $C$ is a convex body in $E^3$, then each component of $\text{cl}(\text{ext } C) \setminus \text{ext } C$ is a subset of a 1-dimensional face of $C$.

Klee raised the question [4] of characterizing the family $\chi$ of all subsets $K$ of the 2-sphere, $S^2$, for which there is a convex body $C$ in $E^3$ and a homeomorphism $h$ of $S^2$ onto $\text{bd } C$ such that $h[K] = \text{ext } C$. In [1] it was shown that if $\text{cl } K$ is 0-dimensional, then $K \in \chi$ if and only if $K$ is a $G_\delta$ set. Theorem 1 indicates that this result cannot be substantially generalized. For example, if $K$ is a countable subset of $S^2$, but $\text{cl}[K] \setminus K$ is a circle, then $K \notin \chi$.

Our proof of Theorem 1 depends on an interesting property of the 1-dimensional faces of a convex body in $E^3$. More generally, we consider for any convex body $C$ in $E^d$, $d \geq 3$, the family $\mathcal{E}(C) = \{\text{ri } F | F \text{ is a } (d - 2)\text{-dimensional face of } C\}$, where $\text{ri } F$ denotes the relative interior of $F$. Recall that the relative interior of $F$ is the interior of $F$ relative to the smallest flat containing it. It is easily seen that $\mathcal{E}(C)$ is a family of pairwise disjoint subsets of $\text{bd } C$ and, consequently, the usual quotient topology may be associated with
it. In general, this topology fails to be metrizable; however, we can show the following:

**Theorem 2.** For any convex body \( C \) in \( E^d \), \( \mathcal{L}(C) \) is paracompact.

This may be compared to the easily observed fact that the quotient topology for the family \( \{ \text{ri} F | F \text{ is a } (d - 1) \text{-dimensional face of } C \} \) is countable and discrete.

We assume in the remainder of the paper that \( C \) is a convex body in \( E^d \), \( d \geq 3 \), \( X = \bigcup \{ A | A \in \mathcal{L}(C) \} \), and \( i : X \to \mathcal{L}(C) \) is defined by \( x \in i(x) \). Thus the quotient topology on \( \mathcal{L}(C) \) is the largest topology which makes \( i \) continuous. Let \( Y \) be the set of all points in \( \text{bd} C \) which do not lie in the relative interior of some \( (d - 1) \)-dimensional face of \( C \). Clearly \( Y \) is a closed set containing \( X \). Moreover, \( X \) is an \( F_\sigma \) set since \( X = \bigcup_{n=1}^{\infty} K_n \) where \( K_n \) is the set of all points which are the center of some closed \( (d - 2) \)-dimensional ball of radius \( 1/n \) contained in \( Y \). Any limit point of \( K_n \) also has this property; hence \( K_n \) is closed.

For any \( S \subseteq E^d \) and \( \epsilon > 0 \), let \( N(S, \epsilon) \) be the open set of all points that lie within \( \epsilon \) of \( S \).

**Lemma 1.** For each \( A \in \mathcal{L}(C) \) and \( \epsilon > 0 \), there is a closed neighborhood of \( A \) in \( \mathcal{L}(C) \) whose members are contained in \( N(A, \epsilon) \).

**Proof.** An open subset \( U \) of \( \text{bd} C \) will be called \( L \)-open if for each \( B \in \mathcal{L}(C) \), \( U \cap B \neq \emptyset \) implies that \( B \subseteq U \). In this case \( \mathcal{U} = \{ B \in \mathcal{L}(C) | U \cap B \neq \emptyset \} \) is open in \( \mathcal{L}(C) \). Let \( K = \text{bd} C \setminus N(A, \epsilon) \) and \( a_1 \in A \). Since \( K \) is compact, it suffices to show that for each \( x \in K \), \( a_1 \) and \( x \) have disjoint \( L \)-open neighborhoods.

Let \( a_2 \in K \) and let \( F \) be the smallest face of \( C \) containing both \( a_1 \) and \( a_2 \). Since \( a_1 \) is in the relative interior of a \( (d - 2) \)-dimensional face, either \( F = C \) or \( F \) has dimension \( d - 1 \). Let \( M \) be the subspace of codimension 1 which is perpendicular to the line through \( a_1 \) and \( a_2 \) and let \( \pi \) be the orthogonal projection of \( E^d \) onto \( M \). Thus \( \pi(a_1) = \pi(a_2) \). Choose some point \( b \) on the line \( M^\perp \) other than the origin and for \( m \in \pi[C] \) define the functions \( f_i(m) = \inf\{ r \in \mathbb{R} | m + rb \in C \} \) and \( f_2(m) = \sup\{ r \in \mathbb{R} | m + rb \in C \} \). We may assume \( a_1 \) is in the graph of \( f_1 \). Clearly \( f_1 \) is convex and \( f_2 \) is concave.

Let \( U_i = \{(x, f_i(x)) | x \in \text{ri} \pi[C] \} \); then \( U_1 \) and \( U_2 \) are disjoint \( L \)-open sets. If \( F = C \), then \( a_1 \in U_i \) and we are done. If \( F \neq C \), let \( V_i = \{(x, f_i(x)) | x \in \text{ri} \pi[F] \} \). Choose a point \( q \in \text{ri} F \), let \( W_i \) be the convex hull of \( V_i \cup \{ q \} \), and let \( V' = V_i \cup \text{ri} W_i \). Then \( U_i = U_i \cup V_i \) is an \( L \)-open set containing \( a_i \) and \( U'_i \), \( U_2 \) are disjoint.

**Proof of Theorem 2.** Lemma 1 implies that \( \mathcal{L}(C) \) is Hausdorff and the second countability of \( E^d \) implies \( \mathcal{L}(C) \) is Lindelöf. Therefore, by a theorem of Morita [2, p. 174], it is sufficient to show \( \mathcal{L}(C) \) regular in order to show that it is paracompact. Let \( \mathcal{K} \) be any closed subfamily of \( \mathcal{L}(C) \) and \( A \in \mathcal{L}(C) \setminus \mathcal{K} \).

We exhibit a closed neighborhood of \( \mathcal{K} \) not containing \( A \).

Since \( X \) is \( \sigma \)-compact and \( i : X \to \mathcal{L}(C) \) is continuous, \( \mathcal{L}(C) \) is also \( \sigma- \)
compact. Hence \( \mathcal{K} \) is the union of some sequence \( \mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \cdots \) of compact subfamilies of \( \mathcal{E}(C) \). Let \( K_n = \bigcup \{B \mid B \in \mathcal{K}_n\} \). It follows from Lemma 1 and the compactness of \( \mathcal{K}_n \) that \( K_n \) has a closed neighborhood \( \mathcal{W}_n \) contained in \( N(K_n, 1/n) \) such that \( A \notin \mathcal{W}_n \). Let \( \mathcal{W} = \bigcup_{n=1}^\infty \mathcal{W}_n \), \( W_n = \bigcup \{B \mid B \in \mathcal{W}_n\} \), \( W = \bigcup_{n=1}^\infty W_n \), and \( K = \bigcup_{n=1}^\infty K_n \). Clearly, \( \mathcal{W} \) is a neighborhood of \( \mathcal{K}, A \notin \mathcal{W}, \) and \( K_n \subseteq W_n \) for each \( n \).

Since \( \mathcal{W} \) is closed if and only if \( W \) is closed relative to \( X \), suppose that there is an \( x \in (X \cap \text{cl} W) \setminus W \). Because \( W_n \) is closed relative to \( X \) for each \( n \), \( x \) must be in the closure of \( \bigcup_{k=1}^\infty W_k \) for each \( k \). Hence \( x \in N(K, 1/k) \) for each \( k \). Since \( K \) is closed relative to \( X \), \( x \in K \). This is a contradiction since \( K \subseteq W \) and the theorem follows. \( \square \)

**Remark.** A straightforward modification of the above proof shows that, in fact, any subfamily of \( \mathcal{E}(C) \) is paracompact.

**Proof of Theorem 1.** Let \( C \) be a convex body in \( E^3 \) and \( D = \text{cl}(\text{ext} C) \setminus \text{ext} C \); then \( \text{ext} C \) and \( X \) form a partition of the closed set \( Y \) and therefore \( D \subseteq X \). Let \( \mathcal{K} = \{A \in \mathcal{E}(C) \mid A \cap D \neq \emptyset\} \) and \( K = \bigcup \{A \mid A \in \mathcal{K}\} \). Since \( \iota : X \to \mathcal{E}(C) \) is continuous, components of \( D \) are mapped into components of \( \mathcal{K} \). In the remainder of the proof we show that \( \mathcal{K} \) must be totally disconnected and this implies Theorem 1.

Since \( X \) is an \( F_\sigma \), \( D \) is an \( F_\sigma \). Let \( D \) be the union of compact subsets \( D_1, D_2, \ldots \) and let \( \mathcal{D}_n = \{A \in \mathcal{E}(C) \mid A \cap D_n \neq \emptyset\} \). The continuity of \( \iota \) implies that \( \mathcal{D}_n \) is compact and therefore closed since \( \mathcal{E}(C) \) is Hausdorff. Thus each set \( D'_n = \bigcup \{A \mid A \in \mathcal{D}_n\} \) is closed relative to \( X \) and must also be an \( F_\sigma \).

It follows that \( K = \bigcup_{n=1}^\infty D'_n \) is an \( F_\sigma \) set. Let \( K \) be the union of the sequence \( V_1 \subseteq V_2 \subseteq \cdots \) of compact sets.

For each rational number \( r \) and each integer \( i, 1 \leq i \leq 3 \), the set \( \{(x_1, x_2, x_3) \in E^3 \mid x_i = r\} \) is a plane in \( E^3 \). Let \( H_1, H_2, \ldots \) be an enumeration of these planes. If \( A \) is any member of \( \mathcal{K} \), there is an integer \( N \) such that for each \( n \geq N, A \cap V_n \) contains an open line segment. Thus for some \( n \geq N \) there is an \( H_n \) which intersects \( A \cap V_n \) in a single point. Let \( K_n \) be the union of all singleton sets of the form \( A \cap V_n \cap H_n \) where \( A \in \mathcal{K} \). If \( x \in (V_n \cap H_n) \setminus K_n \), then \( x \in A \) for some \( A \in \mathcal{K} \) but \( A \cap V_n \cap H_n \) contains more than one point. In this case \( A \subseteq H_n \). If \( K_n \neq \emptyset \), then \( H_n \cap \text{ri} C \neq \emptyset \) and \( A \) is open relative to \( \text{bd} C \cap H_n \), which implies that \( K_n \) is a closed set. Let \( \mathcal{K}_n = \{A \in \mathcal{K} \mid A \cap K_n \neq \emptyset\} \); then \( \mathcal{K} = \bigcup_{n=1}^\infty \mathcal{K}_n \). Moreover, since \( \iota/K_n \) is one-to-one, \( K_n \) is compact, and \( \mathcal{E}(C) \) is Hausdorff, it follows that \( \mathcal{K}_n \) is homeomorphic to \( K_n \).

For each \( x \in X \), let \( f(x) \) be the smallest face of \( C \) containing \( x \). Clearly, \( f(x) = \text{cl} \iota(x) \). If \( \{a_n\} \) is a sequence in \( X \), then \( \lim f(a_n) = \{x \mid f(a_n) \text{ frequently intersects each neighborhood of } x\} \) and \( \lim f(a_n) = \{x \mid f(a_n) \text{ eventually intersects each neighborhood of } x\} \). The function \( f \) is called upper semicontinuous [resp., lower semicontinuous] at \( a \in X \) if for each sequence \( \{a_n\} \) in \( X \) converging to \( a \), \( \lim f(a_n) \subseteq f(a) \) [resp., \( \lim f(a_n) \supseteq f(a) \)]. A slight modification of a theorem of Klee and Martin [5, p. 6] shows that \( f \) is upper semicontinuous at each point of \( X \).
Suppose $K_n$ contains an open line segment $S$. A theorem of Fort [3, p. 287] implies that $f/S$ is lower semicontinuous, as well as upper semicontinuous, at some point $s \in S$. It follows from this that $X$ is a neighborhood of $\Delta(s) = A$ in $\text{bd} C$. Since this contradicts the assumption that $\text{cl}(\text{ext } C) \cap A \neq \emptyset$, $K_n$ can contain no open line segment. Hence $K_n$ is 0-dimensional.

Suppose $A_1$ and $A_2$ are distinct members of $\mathcal{K}$. For $i = 1, 2$, we construct sequences $\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \cdots$ of closed neighborhoods of $A_i$ in $\mathcal{C}(C)$ such that $K_n$ lies in the interior of $\mathcal{W}_n \cup \mathcal{W}_n^2$ but $\mathcal{W}_n^1 \cap \mathcal{W}_n^2 = \emptyset$. Recall that $\mathcal{C}(C)$ is normal since it is paracompact. Let $\mathcal{W}_0^1$ and $\mathcal{W}_0^2$ be any disjoint closed neighborhoods of $A_1$ and $A_2$, respectively. Assume that $\mathcal{W}_{n-1}^1$ and $\mathcal{W}_{n-1}^2$ have been constructed for some $n \geq 1$. Let $\mathcal{K}_n^i = K_n \cap \mathcal{W}_{n-1}^i$. Since $K_n$ is homeomorphic to the compact, 0-dimensional set $K_n$ and $K_n^1$ and $K_n^2$ are disjoint closed subfamilies of $K_n$, there exist disjoint closed subfamilies $\mathcal{F}_n^1 \supseteq K_n^1$ such that $K_n = \mathcal{W}_n^1 \cup \mathcal{W}_n^2$. Let $\mathcal{W}_n^1$ and $\mathcal{W}_n^2$ be disjoint closed neighborhoods of $\mathcal{W}_{n-1}^1 \cup \mathcal{W}_n^1$ and $\mathcal{W}_{n-1}^2 \cup \mathcal{W}_n^2$, respectively. Let $\mathcal{W}_n$ be the interior of $\bigcup_{n=1}^{\infty} \mathcal{W}_n^i$; then $\mathcal{W}_1$ and $\mathcal{W}_2$ are disjoint neighborhoods of $A_1$ and $A_2$, respectively, and $K \subseteq \mathcal{W}_1 \cup \mathcal{W}_2$. Therefore $A_1$ and $A_2$ belong to different components of $\mathcal{K}$. Since $A_1$ and $A_2$ were arbitrary members of $\mathcal{K}$, $\mathcal{K}$ is totally disconnected. □

References


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