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ON THE FACIAL STRUCTURE OF A CONVEX BODY

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ABSTRACT. The family formed by taking the relative interior of each face of a d-dimensional convex body C is a partition of C. It is shown here that the subfamily consisting of all the $(d-2)$-dimensional sets has a quotient topology which is paracompact and this is used to prove a property of the set of extreme points when $d = 3$.

A convex body in E^d is any closed bounded convex subset C with nonempty interior. A flat is any translate of a subspace. A face of C is a convex subset F which is the intersection of C with a flat such that $C \setminus F$ is also convex. Thus the extreme points of C are exactly the 0-dimensional faces of C. The set of extreme points of C will be denoted by $\operatorname{ext} C$ and for any set A the boundary of A and the closure of A will be denoted by $\operatorname{bd} A$ and $\operatorname{cl} A$, respectively.

It is easily seen that if C is 2-dimensional, then $\operatorname{ext} C$ is a closed set. This need not be true in higher dimensions. In fact, there is a convex body C in E^3 for which $\operatorname{ext} C$ and $\operatorname{bd} C \setminus \operatorname{ext} C$ are both dense in $\operatorname{bd} C$ (see [4, p. 104]). However, we can show the following relationship between $\operatorname{ext} C$ and its closure:

THEOREM 1. *If C is a convex body in E^3, then each component of* $\operatorname{cl}(\operatorname{ext} C) \setminus \operatorname{ext} C$ *is a subset of a* 1*-dimensional face of C.*

Klee raised the question [4] of characterizing the family χ of all subsets K of the 2-sphere, S^2, for which there is a convex body C in E^3 and a homeomorphism h of S^2 onto $\operatorname{bd} C$ such that $h[K] = \operatorname{ext} C$. In [1] it was shown that if $\operatorname{cl} K$ is 0-dimensional, then $K \in \chi$ if and only if K is a G_δ set. Theorem 1 indicates that this result cannot be substantially generalized. For example, if K is a countable subset of S^2, but $\operatorname{cl}[K] \setminus K$ is a circle, then $K \notin \chi$.

Our proof of Theorem 1 depends on an interesting property of the 1-dimensional faces of a convex body in E^3. More generally, we consider for any convex body C in E^d, $d \geqslant 3$, the family $\mathcal{L}(C) = \{\operatorname{ri} F \mid F \text{ is a } (d-2)\text{-dimensional face of } C\}$, where $\operatorname{ri} F$ denotes the relative interior of F. Recall that the relative interior of F is the interior of F relative to the smallest flat containing it. It is easily seen that $\mathcal{L}(C)$ is a family of pairwise disjoint subsets of $\operatorname{bd} C$ and, consequently, the usual quotient topology may be associated with

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it. In general, this topology fails to be metrizable; however, we can show the following:

THEOREM 2. *For any convex body C in E^d, $\mathfrak{L}(C)$ is paracompact.*

This may be compared to the easily observed fact that the quotient topology for the family $\{\operatorname{ri} F | F$ is a $(d-1)$-dimensional face of $C\}$ is countable and discrete.

We assume in the remainder of the paper that C is a convex body in E^d, $d \geqslant 3$, $X = \bigcup\{A | A \in \mathfrak{L}(C)\}$, and ι: $X \to \mathfrak{L}(C)$ is defined by $x \in \iota(x)$. Thus the quotient topology on $\mathfrak{L}(C)$ is the largest topology which makes ι continuous. Let Y be the set of all points in $\operatorname{bd} C$ which do not lie in the relative interior of some $(d-1)$-dimensional face of C. Clearly Y is a closed set containing X. Moreover, X is an F_σ set since $X = \bigcup_{n=1}^{\infty} K_n$ where K_n is the set of all points which are the center of some closed $(d-2)$-dimensional ball of radius $1/n$ contained in Y. Any limit point of K_n also has this property; hence K_n is closed.

For any $S \subseteq E^d$ and $\varepsilon > 0$, let $N(S, \varepsilon)$ be the open set of all points that lie within ε of S.

LEMMA 1. *For each $A \in \mathfrak{L}(C)$ and $\varepsilon > 0$, there is a closed neighborhood of A in $\mathfrak{L}(C)$ whose members are contained in $N(A, \varepsilon)$.*

PROOF. An open subset U of $\operatorname{bd} C$ will be called L-open if for each $B \in \mathfrak{L}(C)$, $U \cap B \neq \varnothing$ implies that $B \subseteq U$. In this case $\mathfrak{U} = \{B \in \mathfrak{L}(C) | U \cap B \neq \varnothing\}$ is open in $\mathfrak{L}(C)$. Let $K = \operatorname{bd} C \setminus N(A, \varepsilon)$ and $a_1 \in A$. Since K is compact, it suffices to show that for each $x \in K$, a_1 and x have disjoint L-open neighborhoods.

Let $a_2 \in K$ and let F be the smallest face of C containing both a_1 and a_2. Since a_1 is in the relative interior of a $(d-2)$-dimensional face, either $F = C$ or F has dimension $d-1$. Let M be the subspace of codimension 1 which is perpendicular to the line through a_1 and a_2 and let π be the orthogonal projection of E^d onto M. Thus $\pi(a_1) = \pi(a_2)$. Choose some point b on the line $M^\perp$ other than the origin and for $m \in \pi[C]$ define the functions $f_1(m) = \inf\{r \in \mathbf{R} | m + rb \in C\}$ and $f_2(m) = \sup\{r \in \mathbf{R} | m + rb \in C\}$. We may assume a_i is in the graph of f_i. Clearly f_1 is convex and f_2 is concave.

Let $U_i = \{(x, f_i(x)) | x \in \operatorname{ri} \pi[C]\}$; then U_1 and U_2 are disjoint L-open sets. If $F = C$, then $a_i \in U_i$ and we are done. If $F \neq C$, let $V_i = \{(x, f_i(x)) | x \in \operatorname{ri} \pi[F]\}$. Choose a point $q \in \operatorname{ri} F$, let W_i be the convex hull of $V_i \cup \{q\}$, and let $V'_i = V_i \cup \operatorname{ri} W_i$. Then $U'_i = U_i \cup V'_i$ is an L-open set containing a_i and U'_1, U'_2 are disjoint.□

PROOF OF THEOREM 2. Lemma 1 implies that $\mathfrak{L}(C)$ is Hausdorff and the second countability of E^d implies $\mathfrak{L}(C)$ is Lindelöf. Therefore, by a theorem of Morita [2, p. 174], it is sufficient to show $\mathfrak{L}(C)$ regular in order to show that it is paracompact. Let $\mathfrak{K}$ be any closed subfamily of $\mathfrak{L}(C)$ and $A \in \mathfrak{L}(C) \setminus \mathfrak{K}$. We exhibit a closed neighborhood of $\mathfrak{K}$ not containing A.

Since X is σ-compact and ι: $X \to \mathfrak{L}(C)$ is continuous, $\mathfrak{L}(C)$ is also σ-

compact. Hence $\mathcal{K}$ is the union of some sequence $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \cdots$ of compact subfamilies of $\mathcal{L}(C)$. Let $K_n = \bigcup\{B|B \in \mathcal{K}_n\}$. It follows from Lemma 1 and the compactness of $\mathcal{K}_n$ that $\mathcal{K}_n$ has a closed neighborhood $\mathcal{W}_n$ contained in $N(K_n, 1/n)$ such that $A \notin \mathcal{W}_n$. Let $\mathcal{W} = \bigcup_{n=1}^{\infty}\mathcal{W}_n$, $W_n = \bigcup\{B|B \in \mathcal{W}_n\}$, $W = \bigcup_{n=1}^{\infty} W_n$, and $K = \bigcup_{n=1}^{\infty} K_n$. Clearly, $\mathcal{W}$ is a neighborhood of $\mathcal{K}$, $A \notin \mathcal{W}$, and $K_n \subseteq W_n$ for each n.

Since $\mathcal{W}$ is closed if and only if W is closed relative to X, suppose that there is an $x \in (X \cap \operatorname{cl} W)\backslash W$. Because W_n is closed relative to X for each n, x must be in the closure of $\bigcup_{n=k}^{\infty} W_n$ for each k. Hence $x \in N(K, 1/k)$ for each k. Since K is closed relative to X, $x \in K$. This is a contradiction since $K \subseteq W$ and the theorem follows.□

REMARK. A straightforward modification of the above proof shows that, in fact, any subfamily of $\mathcal{L}(C)$ is paracompact.

PROOF OF THEOREM 1. Let C be a convex body in E^3 and $D = \operatorname{cl}(\operatorname{ext} C)\backslash \operatorname{ext} C$; then $\operatorname{ext} C$ and X form a partition of the closed set Y and therefore $D \subseteq X$. Let $\mathcal{K} = \{A \in \mathcal{L}(C)|A \cap D \neq \varnothing\}$ and $K = \bigcup\{A|A \in \mathcal{K}\}$. Since $\iota\colon X \to \mathcal{L}(C)$ is continuous, components of D are mapped into components of $\mathcal{K}$. In the remainder of the proof we show that $\mathcal{K}$ must be totally disconnected and this implies Theorem 1.

Since X is an F_σ, D is an F_σ. Let D be the union of compact subsets $D_1, D_2, \ldots$ and let $\mathcal{D}_n = \{A \in \mathcal{L}(C)|A \cap D_n \neq \varnothing\}$. The continuity of ι implies that $\mathcal{D}_n$ is compact and therefore closed since $\mathcal{L}(C)$ is Hausdorff. Thus each set $D_n' = \bigcup\{A|A \in \mathcal{D}_n\}$ is closed relative to X and must also be an F_σ. It follows that $K = \bigcup_{n=1}^{\infty} D_n'$ is an F_σ set. Let K be the union of the sequence $V_1 \subseteq V_2 \subseteq \cdots$ of compact sets.

For each rational number r and each integer i, $1 \leqslant i \leqslant 3$, the set $\{(x_1, x_2, x_3) \in E^3|x_i = r\}$ is a plane in E^3. Let $H_1, H_2, \ldots$ be an enumeration of these planes. If A is any member of $\mathcal{K}$, there is an integer N such that for each $n \geqslant N$, $A \cap V_n$ contains an open line segment. Thus for some $n \geqslant N$ there is an H_n which intersects $A \cap V_n$ in a single point. Let K_n be the union of all singleton sets of the form $A \cap V_n \cap H_n$ where $A \in \mathcal{K}$. If $x \in (V_n \cap H_n)\backslash K_n$, then $x \in A$ for some $A \in \mathcal{K}$ but $A \cap V_n \cap H_n$ contains more than one point. In this case $A \subseteq H_n$. If $K_n \neq \varnothing$, then $H_n \cap \operatorname{ri} C \neq \varnothing$ and A is open relative to $\operatorname{bd} C \cap H_n$, which implies that K_n is a closed set. Let $\mathcal{K}_n = \{A \in \mathcal{K}|A \cap K_n \neq \varnothing\}$; then $\mathcal{K} = \bigcup_{n=1}^{\infty}\mathcal{K}_n$. Moreover, since ι/K_n is one-to-one, K_n is compact, and $\mathcal{L}(C)$ is Hausdorff, it follows that $\mathcal{K}_n$ is homeomorphic to K_n.

For each $x \in X$, let $f(x)$ be the smallest face of C containing x. Clearly, $f(x) = \operatorname{cl} \iota(x)$. If $\{a_n\}$ is a sequence in X, then $\overline{\lim} f(a_n) = \{x|f(a_n)$ frequently intersects each neighborhood of $x\}$ and $\underline{\lim} f(a_n) = \{x|f(a_n)$ eventually intersects each neighborhood of $x\}$. The function f is called upper semicontinuous [resp., lower semicontinuous] at $a \in X$ if for each sequence $\{a_n\}$ in X converging to a, $\overline{\lim} f(a_n) \subseteq f(a)$ [resp., $\underline{\lim} f(a_n) \supseteq f(a)$]. A slight modification of a theorem of Klee and Martin [5, p. 6] shows that f is upper semicontinuous at each point of X.

Suppose K_n contains an open line segment S. A theorem of Fort [3, p. 287] implies that f/S is lower semicontinuous, as well as upper semicontinuous, at some point $s \in S$. It follows from this that X is a neighborhood of $\iota(s) = A$ in bd C. Since this contradicts the assumption that $\text{cl}(\text{ext}\, C) \cap A \neq \emptyset$, K_n can contain no open line segment. Hence K_n is 0-dimensional.

Suppose A_1 and A_2 are distinct members of $\mathcal{K}$. For $i = 1, 2$, we construct sequences $\mathcal{W}_0^i \subseteq \mathcal{W}_1^i \subseteq \cdots$ of closed neighborhoods of A_i in $\mathcal{L}(C)$ such that $\mathcal{K}_n$ lies in the interior of $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ but $\mathcal{W}_n^1 \cap \mathcal{W}_n^2 = \emptyset$. Recall that $\mathcal{L}(C)$ is normal since it is paracompact. Let $\mathcal{W}_0^1$ and $\mathcal{W}_0^2$ be any disjoint closed neighborhoods of A_1 and A_2, respectively. Assume that $\mathcal{W}_{n-1}^1$ and $\mathcal{W}_{n-1}^2$ have been constructed for some $n \geqslant 1$. Let $\mathcal{K}_n^i = \mathcal{K}_n \cap \mathcal{W}_{n-1}^i$. Since $\mathcal{K}_n$ is homeomorphic to the compact, 0-dimensional set K_n and $\mathcal{K}_n^1$ and $\mathcal{K}_n^2$ are disjoint closed subfamilies of $\mathcal{K}_n$, there exist disjoint closed subfamilies $\mathcal{P}_n^i \supseteq \mathcal{K}_n^i$ such that $\mathcal{K}_n = \mathcal{P}_n^1 \cup \mathcal{P}_n^2$. Let $\mathcal{W}_n^1$ and $\mathcal{W}_n^2$ be disjoint closed neighborhoods of $\mathcal{W}_{n-1}^1 \cup \mathcal{P}_n^1$ and $\mathcal{W}_{n-1}^2 \cup \mathcal{P}_n^2$, respectively. Let $\mathcal{W}_i$ be the interior of $\bigcup_{n=1}^{\infty} \mathcal{W}_n^i$; then $\mathcal{W}_1$ and $\mathcal{W}_2$ are disjoint neighborhoods of A_1 and A_2, respectively, and $\mathcal{K} \subseteq \mathcal{W}_1 \cup \mathcal{W}_2$. Therefore A_1 and A_2 belong to different components of $\mathcal{K}$. Since A_1 and A_2 were arbitrary members of $\mathcal{K}$, $\mathcal{K}$ is totally disconnected.□

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