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MONIC AND MONIC FREE IDEALS IN A POLYNOMIAL SEMIRING IN SEVERAL VARIABLES

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ABSTRACT. The study of monic and monic free ideals in a polynomial semiring $S[x]$, where S is a commutative semiring with an identity, is extended to $S[x_1, x_2, \ldots, x_n]$. Structure theorems are given for monic, monic free, and monic free k-ideals in $S[x_1, x_2, \ldots, x_n]$. It is shown that each monic free k-ideal in $S[x_1, \ldots, x_n]$, S a strict semiring, is the sum of a finite number of ideals B_i such that each B_i is the union of a proper infinite ascending chain of ideals.

1. **Introduction.** Structure theorems for monic and monic free ideals in a polynomial semiring $S[x]$, where S is a strict semiring, were developed in [1]. In this paper, these theorems will be generalized to polynomial semirings in several variables and a decomposition theorem will be given for monic free k-ideals.

2. **Fundamentals.** The concept of a ring is derived from, and is a generalization of, the algebraic aspects of the set of the set of ordinary integers. Consequently, it should not be surprising that the concept of a semiring is a generalization of the algebraic aspects of the set of nonnegative integers.

2.1. DEFINITION. A nonempty set S together with two binary operations $(+)$ and $(\cdot)$ is said to be a semiring if $(S, +)$ is an abelian semigroup with a zero, $(S, \cdot)$ is a semigroup, and $(\cdot)$ is distributive over $(+)$ from both the left and the right. The semiring S is said to be strict if $a \in S$, $b \in S$, and $a + b = 0$ imply $a = b = 0$. If $(S, \cdot)$ is a commutative semigroup, then S will be called a commutative semiring. Throughout this paper S will be a commutative semiring.

2.2. DEFINITION. An ideal I in a semiring S will be called a k-ideal if $a \in I$, $b \in S$ and $a + b \in I$ imply $b \in I$.

Let S be a semiring with an identity and $x_1, x_2, \ldots, x_n$ be indeterminates which commute with each other and with each element of S. It is clear that $S[x_1, x_2, \ldots, x_n]$ is a semiring and that a typical element in $S[x_1, x_2, \ldots, x_n]$ is of the form $\sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ with $a_{i_1 \cdots i_n} \in S$.

The following notation will be used throughout this paper. Let $S(k, n)$ denote the set of sequences of length n from the integers 0, 1, 2, $\ldots$, k, and Φ

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$= x_1 x_2 \cdots x_n$. For $\alpha = \{i_1, \ldots, i_n\} \in S(k,n)$, let

$$a_\alpha = a_{i_1 \cdots i_n} \quad \text{and} \quad \Phi^\alpha = x_1^{i_1} \cdots x_n^{i_n}.$$

With this notation, the polynomial $\sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$ can be written $\sum a_\alpha \Phi^\alpha$, $\alpha \in S(k,n)$. Also let $S(\infty, n)$ denote the set of sequences of length n from the set of nonnegative integers. It is clear that $S(k,n) \subset S(\infty, n)$.

2.3. DEFINITION. An ideal M in $S[x_1, \ldots, x_n]$ will be called monic if $f(x_1, \ldots, x_n) = \sum a_\alpha \Phi^\alpha \in M$ implies that each $a_\alpha \Phi^\alpha \in M$.

2.4. DEFINITION. An ideal F in $S[x_1, \ldots, x_n]$ will be called monic free if M is any monic ideal such that $M \subset F$, then $M = \{0\}$.

If $f(x_1, x_2, \ldots, x_n) = \sum a_\alpha \Phi^\alpha \in S[x_1, x_2, \ldots, x_n]$, then for any x_i, one can write

$$f(x_1, x_2, \ldots, x_n) = \sum_k P_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x_i^k,$$

where $P_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in S[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Using this observation, one can slightly weaken the concepts of monic and monic free ideals in $S[x_1, x_2, \ldots, x_n]$.

2.5. DEFINITION. An ideal M in $S[x_1, \ldots, x_n]$ will be called monic in x_i if $f(x_1, \ldots, x_n) = \sum_k P_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x_i^k \in M$ implies that $P_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x_i^k \in M$ for each k.

2.6. DEFINITION. An ideal F in $S[x_1, \ldots, x_n]$ will be called monic free in x_i if M is an ideal such that $M \subset F$ and M is monic in x_i, then $M = \{0\}$.

2.7. EXAMPLES. Let Z^+ be the nonnegative integers and $Z^+[x,y]$ the semiring of polynomials over Z^+ in x and y. Let $A = (x + y)$, $B = (xy)$, and $C = (1 + y)$ be the ideals in $Z^+[x,y]$ generated by $x + y$, xy, and $1 + y$ respectively. The ideal A is monic free in both x and y. For if $M \neq 0$ is monic in x and $M \subset A$, then there is a nonzero polynomial $P(y) = \sum_0^n a_i y^i$ such that $P(y)x^k \in M \subset A$. Consequently, $P(y)x^k = f(x,y)(x + y)$ for some $f(x,y) \in Z^+[x,y]$. Replacing x by -1 and y by $+1$ yields $P(1) = 0$. However, $P(1) = a_n + a_{n-1} + \cdots + a_0 = 0$ and Z^+ being a strict semiring assures that $a_n = a_{n-1} = \cdots = a_0 = 0$. Consequently, $P(y) = 0$, a contradiction. Therefore $M = 0$ and A is monic free in x. Similarly A is monic free in y. It is easy to see that the ideal $B = (xy)$ is monic in both x and y. In a manner similar to the above, it can be shown that the ideal $C = (1 + y)$ is monic free in y. However, C is monic in x. For if $\sum P_k(y)x^k \in C$, then $\sum P_k(y)x^k = f(x,y)(1 + y)$ for some $f(x,y) \in Z^+[x,y]$. Hence, any coefficient of x^k has $(1 + y)$ as a factor. Thus, $P_k(y)x^k$ has $(1 + y)$ as a factor. Thus, $P_k(y)x^k \in C$ for each k and C is monic in x.

These examples point out clearly that an ideal in $S[x_1, \ldots, x_n]$ may be monic in each variable, monic free in each variable, or monic in some variables while monic free in others.

3. **Monic ideals in** $S[x_1, \ldots, x_n]$. A collection of sets $G = \{A_\alpha \mid \alpha \in S(\infty, n)$ and n is fixed$\}$ will be called a generalized ascending chain if A_α and A_β are

two sets in G such that $\alpha = \{i_1, \ldots, i_n\}$, $\beta = \{j_1, \ldots, j_n\}$, and $i_k \leqslant j_k$ for each k, then $A_\alpha \subseteq A_\beta$. For example, the collection of sets $G = \{A_\alpha | \alpha \in S(\infty, n)$, where $A_\alpha = \{0, 1, 2, \ldots, \max\{\alpha\}\}$, is a generalized ascending chain of sets. If each A_α is an ideal, then G is called a generalized ascending chain of ideals.

Let $G = \{A_\alpha | \alpha \in S(\infty, n)\}$ be a generalized ascending chain of ideals in a semiring S and $G^* = \{\sum a_\alpha \Phi^\alpha \in S[x_1, \ldots, x_n] | a_\alpha \in A_\alpha\}$.

3.1. THEOREM. *G^* is a monic ideal in $S[x_1, \ldots, x_n]$.*

PROOF. Let $f = \sum a_\alpha \Phi^\alpha$ and $g = \sum b_\alpha \Phi^\alpha$ be elements of G^* and $h = \sum c_\beta \Phi^\beta \in S[x_1, \ldots, x_n]$. Also let $\alpha = \{i_1, \ldots, i_n\}$ and $\beta = \{r_1, \ldots, r_n\}$. Since A_α is an ideal in S, it follows that $a_\alpha + b_\alpha \in A_\alpha$ and consequently, $f + g \in G^*$. Now $hf = \sum d_\omega \Phi^\omega$ where $\omega = \{j_1, \ldots, j_n\}$ and $d_\omega = \sum c_\beta a_\alpha$ for $\omega = \beta + \alpha$. Clearly, $j_k = r_k + i_k$ and $i_k \leqslant j_k$ for each k and it follows that $a_\alpha \in A_\omega$ since G is a generalized ascending chain of ideals. Thus, $d_\omega \in A_\omega$ and $hf \in G^*$. It follows from the definition of G^* that G^* is monic.

Next let B be an ideal in $S[x_1, \ldots, x_n]$. For each $\alpha \in S(\infty, n)$, let $B_\alpha = \{a \in S |$ there is an $f \in B$ such that $a\Phi^\alpha$ is a term of $f\}$ and $H = \{B_\alpha\}$.

3.2. THEOREM. *If B is an ideal in $S[x_1, \ldots, x_n]$, then H is a generalized ascending chain of ideals in S.*

PROOF. For $a \in B_\alpha$ and $b \in B_\alpha$ there are polynomials f and g in B such that $a\Phi^\alpha$ and $b\Phi^\alpha$ are terms of f and g respectively. Now $f + g \in B$ and $(a + b)\Phi^\alpha$ is a term of $f + g$ and it follows that $a + b \in B_\alpha$. If $c \in S$, then $cf \in B$ and $ca\Phi^\alpha$ is a term of cf. Consequently, $ca \in B_\alpha$. Next, suppose that B_α and B_β are such that $\alpha = \{i_1, \ldots, i_n\}$, $\beta = \{r_1, \ldots, r_n\}$, $i_k \leqslant r_k$ for each k and $b \in B_\alpha$. Then there is a polynomial $f \in B$ such that $b\Phi^\alpha$ is a term of f. Let $\beta - \alpha = \{r_1 - i_1, \ldots, r_n - i_n\}$. Then, $\Phi^{\beta-\alpha} f \in B$ with $\Phi^{\beta-\alpha} b\Phi^\alpha = b\Phi^\beta$ as one of its terms. Thus, $b \in B_\beta$ and it follows that $B_\alpha \subseteq B_\beta$ and H is a generalized ascending chain of ideals in S.

For each ideal B in $S[x_1, \ldots, x_n]$ the above theorem assures the existence of a generalized ascending chain of ideals $\{B_\alpha\}$ in S. Let $B^* = \{\sum a_\alpha \Phi^\alpha \in S[x_1, \ldots, x_n] | a_\alpha \in B_\alpha\}$.

3.3. THEOREM. *An ideal B in $S[x_1, \ldots, x_n]$ is monic if and only if $B = B^*$.*

PROOF. If $B = B^*$, Theorem 3.1 assures that B is a monic ideal. Suppose that B is monic and $f = \sum a_\alpha \Phi^\alpha \in B^*$. Then $a_\alpha \in B_\alpha$ and there is a polynomial $g \in B$ such that $a_\alpha \Phi^\alpha$ is a term of g. Now B being monic assures that $a_\alpha \Phi^\alpha \in B$ and it follows that $\sum a_\alpha \Phi^\alpha = f \in B$. Hence $B^* \subseteq B$. Also if B is monic it is clear that $B \subseteq B^*$. Therefore $B = B^*$.

3.4. COROLLARY. *A monic ideal B in $S[x_1, \ldots, x_n]$ is a k-ideal if and only if each B_α is a k-ideal.*

PROOF. If B is a monic k-ideal in $S[x_1, \ldots, x_n]$, then $B = B^*$. Let

$a \in B_\alpha$, $b \in S$ and $a + b \in B_\alpha$. Then $a\Phi^\alpha \in B$, $(a + b)\Phi^\alpha = a\Phi^\alpha + b\Phi^\alpha \in B$ and $b\Phi^\alpha \in B$ since B is a k-ideal. Consequently, $b \in B_\alpha$ and B_α is a k-ideal. Conversely, suppose that each B_α is a k-ideal, $f = \sum a_\alpha \Phi^\alpha \in B$, $g = \sum b_\alpha \Phi^\alpha \in S[x_1, \ldots, x_n]$, and $f + g \in B$. Then $a_\alpha \in B_\alpha$, $a_\alpha + b_\alpha \in B_\alpha$ and it follows that $b_\alpha \in B_\alpha$. Consequently, $b_\alpha \Phi^\alpha \in B$ and $\sum b_\alpha \Phi^\alpha = g \in B$. Therefore B is a k-ideal.

It may happen that an ideal is monic in some variables but not in others as Example 2.7 shows. But if an ideal is monic in each variable one has the following.

3.5. THEOREM. *An ideal B in $S[x_1, \ldots, x_n]$ is monic if and only if B is monic in each x_i.*

PROOF. Clearly B is monic in each variable if B is monic. Suppose B is monic in each variable x_i and $f = \sum a_\alpha \Phi^\alpha \in B$, $\alpha \in S(\infty, n)$. To show that $a_\alpha \Phi^\alpha \in B$, let $\alpha = \{i_1, \ldots, i_n\}$. Recall that f can be written in the form $f = \sum_k [P_k(x_1, \ldots, x_{n-1})]x_n^k$, where $P_k(x_1, \ldots, x_{n-1}) \in S[x_1, \ldots, x_{n-1}]$ for each k. Now B monic in x_n assures that $[P_k(x_1, \ldots, x_{n-1})]x_n^k \in B$ for each k. In particular for $k = i_n$,

$$[P_{i_n}(x_1, \ldots, x_{n-1})]x_n^{i_n} = (x_n^{i_n})P_{i_n}(x_1, \ldots, x_{n-1}) \in B.$$

Since $a_\alpha \Phi^\alpha$ is a term of f, it follows that $a_\alpha x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}$ is a term of $P_{i_n}(x_1, \ldots, x_{n-1})$. Now $P_{i_n}(x_1, \ldots, x_{n-1}) = \sum_k [P_{i_n k}(x_1, \ldots, x_{n-2})]x_{n-1}^k$, where $P_{i_n k}(x_1, \ldots, x_{n-2}) \in S[x_1, \ldots, x_{n-2}]$ for each k. Since B is monic in x_{n-1} and $(x_n^{i_n})P_{i_n}(x_1, \ldots, x_{n-1}) \in B$, it follows that

$$(x_n^{i_n})[P_{i_n k}(x_1, \ldots, x_{n-2})]x_{n-1}^k \in B$$

for each k. Letting $k = i_{n-1}$, one obtains that $a_\alpha x_1^{i_1} \cdots x_{n-2}^{i_{n-2}}$ is a term of $P_{i_n i_{n-1}}(x_1, \ldots, x_{n-2})$. Continuing in this manner, one obtains a polynomial $P_{i_n \cdots i_2}(x_1) = \sum_k (P_{i_n \cdots i_2 k})x_1^k$ such that $P_{i_n \cdots i_2 k} \in S$ for each k, $(x_2^{i_2} \cdots x_n^{i_n})P_{i_n \cdots i_2}(x_1) \in B$, and $a_\alpha x_1^{i_1}$ is a term of $P_{i_n \cdots i_2}(x_1)$. Since B is monic in x_1, it follows that $(x_2^{i_2} \cdots x_n^{i_n})(P_{i_n \cdots i_2 k})x_1^k \in B$ for each k. In particular for $k = i_1$,

$$(x_2^{i_2} \cdots x_n^{i_n})(P_{i_n \cdots i_1})x_1^{i_1} = (P_{i_n \cdots i_1})(x_1^{i_1} \cdots x_n^{i_n}) \in B.$$

From $a_\alpha x_1^{i_1}$ being a term of $P_{i_n \cdots i_2}(x_1)$ and $P_{i_n \cdots i_1} \in S$, it follows that $P_{i_n \cdots i_1} = a_\alpha$. Consequently, $(P_{i_n \cdots i_1})(x_1^{i_1} \cdots x_n^{i_n}) = a_\alpha \Phi^\alpha \in B$ and B is monic.

In Example 2.7, the ideal $B = (xy)$ is monic in both x and y and consequently, is a monic ideal.

4. **Monic free ideals in $S[x_1, \ldots, x_n]$.** Monic ideals in $S[x_1, \ldots, x_n]$ can be characterized in terms of each variable. A similar characterization for monic free ideals is not valid, as the ideal $C = (1 + y)$ in Example 2.7 shows. For C

is monic free in $Z^+[x,y]$, but C is not monic free in x. To find a suitable characterization of monic free ideals, attention is now turned to the basis elements. Suppose B is an ideal in $S[x_1,\ldots,x_n]$ with a basis U and $X = \{x_1,x_2,\ldots,x_n\}$. If $P = \{x_{\alpha_1},\ldots,x_{\alpha_r}\}$ is a subset of X, let $X_P = \{g(x_{\alpha_1},\ldots,x_{\alpha_r}) | g(x_{\alpha_1},\ldots,x_{\alpha_r}) \in U\}$ if $P \neq \varnothing$, and $X_P = \{c \in B | c \in U\}$ if $P = \varnothing$. Note that the X_P's partition U. For clearly $X_P \cap X_{P'} = \varnothing$ if $P \neq P'$ and $U = \bigcup X_P$, $P \subset X$. Next, let $B_P = (X_P)$ be the ideal generated by X_P. Then $B = \sum B_P$, $P \subset X$, is a decomposition of B into at most 2^n ideals with nonintersecting bases.

4.1. THEOREM. *Let S be a strict semiring. An ideal B in $S[x_1,\ldots,x_n]$ is monic free if and only if each nonempty B_P is monic free.*

PROOF. If B is monic free and M is a monic ideal such that $M \subset B_P \subset B$, then clearly $M = 0$ and B_P is monic free. Conversely, suppose each nonempty B_P is monic free and M is a nonzero monic ideal in B. Let $\Phi = x_{\alpha_1}\cdots x_{\alpha_r}$ and $\beta = \{i_1,\ldots,i_r\}$. Then M contains a term of the form $a\Phi^\beta$ with $a \neq 0$. Let $P_\alpha = \{x_{\alpha_1},\ldots,x_{\alpha_r}\}$. Then $P_\alpha \subset X$, $P_\alpha \neq \varnothing$, and S being a strict semiring assures that $a\Phi^\beta \in B_{P_\alpha}$. Otherwise, $a\Phi^\beta$ could possibly be written as a linear combination of polynomials from other B_P. Consequently, the ideal $(a\Phi^\beta)$ generated by $a\Phi^\beta$ is a nonzero monic ideal in B_{P_α}. But this contradicts the assumption that each nonempty B_P is monic free. Therefore $M = 0$ and B is monic free.

At this point, the following result from [1] is restated for use here.

4.2. LEMMA. *Let S be a strict semiring. If F is a monic free k-ideal in $S[x]$, then $F = \bigcup F_\alpha$, where $\{F_\alpha\}$ is a proper ascending chain of ideals.*

Since it is well known that

$$S[x_1,\ldots,x_n] = S[x_1,\ldots,x_{n-1}][x_n] = S[x_1,\ldots,x_n],$$

Lemma 4.2 can be extended to $S[x_1,\ldots,x_n]$ if it can be shown that $S[x_1,\ldots,x_{n-1}]$ is a strict semiring.

4.3. LEMMA. *$S[x]$ is a strict semiring if and only if S is a strict semiring.*

PROOF. Since $S \subset S[x]$, it is clear that if $S[x]$ is a strict semiring, so is S. Conversely, suppose S is a strict semiring and $f = \sum a_i x^i$, $g = \sum b_i x^i \in S[x]$ such that $f + g = \sum (a_i + b_i)x^i = 0$. Clearly $a_i + b_i = 0$ for each i and it follows that $a_i = b_i = 0$. Consequently $f = g = 0$ and $S[x]$ is a strict semiring.

4.4. COROLLARY. *$S[x_1,\ldots,x_n]$ is a strict semiring if and only if S is a strict semiring.*

Now suppose B is a monic free k-ideal in $S[x_1,\ldots,x_n]$. Lemma 4.2 assures that $B = \bigcup C_\alpha$, where $\{C_\alpha\}$ is a proper ascending chain of ideals. From $B_P \subset B$, it follows that

$$B_P = B_P \cap B = B_P \cap \{\bigcup C_\alpha\} = \bigcup\{B_P \cap C_\alpha\},$$

where $\{B_P \cap C_\alpha\}$ is a proper ascending chain of ideals. Since B_P is generated by polynomials in $x_{\alpha_1}, \ldots, x_{\alpha_r}$, it is clear that $B_P \cap C_\alpha$ is also generated by polynomials in $x_{\alpha_1}, \ldots, x_{\alpha_r}$. These remarks prove the following.

4.5. THEOREM. *Let S be a strict semiring and B a monic free k-ideal in $S[x_1, \ldots, x_n]$. Then each nonempty B_P is monic free and $B = \sum B_P$, $P \subset X$, where $B_P = \bigcup E_\alpha$, $\{E_\alpha\}$ being a proper ascending chain of ideals each generated by polynomials in the same variables.*

4.6. EXAMPLE. Consider the integers Z and the nonnegative integers Z^+. Define a mapping η: $Z^+[x,y] \to Z[i]$, $i = \sqrt{-1}$, by $\eta(f(x,y)) = f(i,i)$. That η is a semiring homomorphism and $\ker \eta = F$ is a k-ideal is clear. Now suppose that M is a monic ideal such that $M \subset F$. If $bx^p y^q \in M$, then

$$\eta(bx^p y^q) = b(i)^p(i)^q = b(i)^{p+q} = 0.$$

Hence $b = 0$ and it follows that $M = 0$ and F is monic free. Now let

$$U = \{x^r + y^s \mid |r - s| = 4n + 2\} \cup \{x^r y^s + 1 \mid r + s = 4n + 2\},$$

and $B = (U)$. Note that

$$\eta(x^r + y^s) = i^r + i^s = i^s(i^{r-s} + 1) = i^s(i^{4n+2} + 1) = i^s(-1 + 1) = 0$$

if $r > s$, and

$$\eta(x^r y^s + 1) = i^r i^s + 1 = i^{r+s} + 1 = i^{4n+2} + 1 = -1 + 1 = 0.$$

Hence if $f \in B$, it is clear that $\eta(f) = 0$ and $B \subset F$. Now let $f(x,y) = \sum a_{pq} x^p y^q \in F$. Then $\eta(f(x,y)) = f(i,i) = \sum a_{pq} i^p i^q = 0$. Let $f(x,y) = f_1(x,y) + f_2(x,y)$ where $f_1(x,y)$ has only odd degree terms and $f_2(x,y)$ has only even degree terms. It follows that $f_1(i,i) = 0 = f_2(i,i)$. Now the odd powers of i give only $\pm i$ and the even powers of i give only ± 1. Hence to get different signs on i or 1, the powers must differ by $4n + 2$. Using this and the fact that the coefficients a_{pq} are nonnegative integers, it is straightforward to show, by rearranging terms and factoring, that

$$f(x,y) = \sum (x^r + y^s) g_{rs} + \sum (x^t y^p + 1) h_{tp},$$

where r, s, t and p are nonnegative integers, $g_{rs}, h_{tp} \in Z^+[x,y]$, $|r - s| = 4k + 2$, and $t + p = 4k + 2$. Consequently, $f(x,y) \in B$ and $F \subset B$. Therefore $F = B$ and F is a monic free k-ideal. The fact that the coefficients are nonnegative integers is very important. For example, the polynomial $f(x,y) = x^6 + 2xy^3 + y^2 \in F$ can be written

$$f(x,y) = x^6 + 2xy^3 + y^2 = x^6 + xy^3 + xy^3 + y^2$$
$$= x(x^5 + y^3) + y^2(xy + 1).$$

Clearly $f(x,y) \in B$. On the other hand, a polynomial $f(x,y) = x + x^3 \in F$ can be written $f(x,y) = x + x^3 = x(1 + x^2)$. Here $g_{rs} = x$, $r = 2$ and $s = 0$. Now consider U, the basis for B. Recall that $X = \{x,y\}$. Hence there are four subsets of X, namely, $P_0 = \varnothing$, $P_1 = \{x\}$, $P_2 = \{y\}$ and $P_3 = \{x,y\}$. Clearly $X_{P_0} = \varnothing$ and $B_{P_0} = \varnothing$. By letting $s = 0$, one obtains $X_{P_1} = \{s^r + 1 | r = 4n + 2\}$. Similarly, letting $r = 0$ gives $X_{P_2} = \{y^s + 1 | s = 4n + 2\}$. Consequently,

$$X_{P_3} = \{x^r + y^s | r \neq 0, s \neq 0, \text{ and } |r - s| = 4n + 2\} \cup \{x^r y^s + 1 | r \neq 0, s \neq 0, \text{ and } r + s = 4n + 2\}.$$

Hence $U = X_{P_1} \cup X_{P_2} \cup X_{P_3}$ where $X_{P_i} \cap X_{P_j} = \varnothing$ if $P_i \neq P_j$ and $B = B_{P_1} + B_{P_2} + B_{P_3}$. Since B is a monic free k-ideal, Theorem 4.5 assures that each B_{P_i} can be written as the union of a proper ascending chain of ideals. Let

$$B_{10} = (x^2 + 1),$$

$$B_{11} = (x^6 + 1) + B_{10}, \ldots, B_{1n} = (x^{4n+2} + 1) + B_{1(n-1)}, \ldots.$$

Clearly $B_{P_1} = \sum B_{1i}$. Similarly, letting

$$B_{20} = (y^2 + 1),$$

$$B_{21} = (y^6 + 1) + B_{20}, \ldots, B_{2n} = (y^{4n+2} + 1) + B_{2(n-1)}, \ldots$$

gives $B_{P_2} = \sum B_{2i}$. Now B_{P_3} has basis X_{P_3}. By considering the exponents r and s of the variables x and y to be ordered pairs of positive integers, it is clear that the set C of all such ordered pairs is a proper subset of $N \times N$, N the set of natural numbers. Thus, the diagonal method of counting can be employed to construct sums of ideals B_{ij} such that $\{B_{ij}\}$ is a proper ascending chain of ideals. To do this, let

$$B_{ij} = \begin{cases} (x^i + y^j) & \text{if } |i - j| = 4n + 2, \\ (x^i y^j + 1) & \text{if } i + j = 4n + 2, \\ (x^i + y^j) + (x^i y^j + 1) & \text{if } |i - j| = 4n + 2 \text{ and } i + j = 4k + 2 \\ & \text{for some } n \text{ and } k. \end{cases}$$

Then letting

$$B_{11} = (xy + 1), \qquad B_{13} = (x + y^3) + B_{11},$$

$$B_{24} = ((x^2 + y^4) + (x^2 y^4 + 1)) + B_{13}, \ldots$$

gives $B_{P_3} = \sum\{B_{ij}\}$, where $\{B_{ij}\}$ is a proper ascending chain of ideals.

A method similar to the above can be used in general to construct monic free k-ideals in $Z^+[x_1, x_2, \ldots, x_n]$.

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