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THE EQUIVALENCE OF VARIOUS LIPSCHITZ CONDITIONS ON THE WEIGHTED MEAN OSCILLATION OF A FUNCTION

EDWARD P. LOTKOWSKI AND RICHARD L. WHEEDEN[1]

ABSTRACT. The main result establishes the equivalence of various Lipschitz-like conditions on the weighted mean oscillation over cubes of a function. Previously known results are obtained as special cases.

Introduction. In [3], John and Nirenberg proved that if f is a function of bounded mean oscillation, i.e., if

$$\int_I |f(x) - f_I| dx \leqslant c|I|, \qquad f_I = \frac{1}{|I|}\int_I f(t)\,dt,$$

for all cubes $I \subset R^n$, with c independent of I, then there exist positive constants c_1 and c_2 depending only on n such that

$$|\{x \in I\colon |f(x) - f_I| > \alpha\}| \leqslant c_1 e^{-c_2\alpha/c}|I|$$

for all I and $\alpha > 0$. In [7] Muckenhoupt and Wheeden derived an analogous result for functions of weighted bounded mean oscillation, by which we mean those f with

$$\int_I |f(x) - f_I| dx \leqslant c\int_I w(x)\,dx$$

for all I, where w is nonnegative and satisfies appropriate conditions.

Meyers [5] and Campanato [1] showed independently that the condition

$$\int_I |f(x) - f_I| dx \leqslant c|I|^{1+\varepsilon}$$

for an ε satisfying $0 < \varepsilon \leqslant 1$ is equivalent (after redefining f in a set of measure 0) to Lipschitz continuity of order ε. Two different weighted versions of this result were then obtained by Cuerva [2] and Lotkowski [4]. Our main theorem generalizes and unifies these latter results, as we shall indicate below.

We first list some definitions. If μ is a Borel measure which is positive and

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finite on cubes and f is locally integrable with respect to μ, then $f_{I,d\mu}$ (or simply f_I) will denote the average of f over I with respect to μ:

$$f_I = f_{I,d\mu} = \frac{1}{\mu(I)} \int_I f(x)\, d\mu(x).$$

For $a > 0$, aI will denote the cube concentric with I whose edge length is a times that of I. We shall be concerned only with measures which satisfy the doubling condition

$$\mu(2I) \leqslant c\mu(I) \tag{1}$$

for every I, with c independent of I. It is an easy consequence of (1) that two cubes of equal size whose centers are at a distance bounded by a fixed multiple of their edge length have equivalent μ measure. For $1 < p < \infty$, a nonnegative function u is said to belong to $A_p(d\mu)$ if there is a constant c such that for all I

$$\left(\frac{1}{\mu(I)} \int_I u(x)\, d\mu(x)\right)\left(\frac{1}{\mu(I)} \int_I u(x)^{-1/(p-1)} d\mu(x)\right)^{p-1} \leqslant c. \tag{2}$$

Similarly, u is said to belong to $A_1(d\mu)$ if

$$u_\mu^*(x) \leqslant cu(x) \quad \text{a.e. } (d\mu), \tag{3}$$

with c independent of x, where u_μ^* is the Hardy-Littlewood maximal function of u with respect to $d\mu$:

$$u_\mu^*(x) = \sup\left\{\frac{1}{\mu(I)} \int_I u(t)\, d\mu(t) \colon x \in I\right\}.$$

$F(I)$ will denote a positive function of cubes for which there exist constants α, β and c such that $1 < \alpha < \beta$, $c > 0$ and

$$F(I) \leqslant cF(J) \quad \text{if } I \subset J, \tag{4}$$

$$\alpha F(I) \leqslant F(2I) \leqslant \beta F(I). \tag{5}$$

It is crucial for what follows that α strictly exceed 1. An example of such an F is

$$F(I) = [\mu_1(I)]^{\varepsilon_1} \cdots [\mu_m(I)]^{\varepsilon_m}$$

with μ_j satisfying (1) and $\varepsilon_j > 0$ for $1 \leqslant j \leqslant m$.

We shall often use the same letter c to denote different constants which are independent of x and I. For a function g which is nonzero a.e., g^{-1} will denote the function $1/g$.

Our main result is given by the following theorem.

THEOREM. *Suppose that g is nonnegative and locally integrable with respect to*

μ, that the measure $g d\mu$ satisfies (1), *and that F satisfies* (4) *and* (5). *If $1 \leqslant p < \infty$ and $g^{-1} \in A_p(g\,d\mu)$, then the condition*

$$\int_I |f(x) - f_I| g(x)\, d\mu(x) \leqslant c\mu(I) F(I), \qquad f_I = f_{I,gd\mu}, \tag{6}$$

is equivalent to

$$\int_I \{|f(x) - f_I| g(x)\}^{p'} d\mu(x) \leqslant c\mu(I) F(I)^{p'}, \qquad 1/p + 1/p' = 1. \tag{7}$$

In case $p = 1$, we have $p' = \infty$, and (7) should be interpreted in the L^∞ sense:

$$\operatorname*{ess\,sup}_I \{|f(x) - f_I| g(x)\} \leqslant cF(I), \tag{8}$$

the ess sup being taken with respect to $d\mu$, or equivalently $gd\mu$.

Before proceeding with the proof, we list two important special cases. First, given $\varepsilon > 0$ and a weight function w, we choose

$$\mu(I) = \int_I w(x)\, dx, \qquad F(I) = \left[\int_I w(x)\, dx\right]^\varepsilon, \qquad g(x) = w(x)^{-1},$$

obtaining that for $1 < p < \infty$, the condition

$$\int_I |f(x) - f_I| dx \leqslant c\left[\int_I w(x)\, dx\right]^{1+\varepsilon}, \qquad f_I = f_{I,dx}, \tag{9}$$

is equivalent to

$$\int_I |f(x) - f_I|^{p'} w(x)^{-1/(p-1)} dx \leqslant c\left[\int_I w(x)\, dx\right]^{1+\varepsilon p'}, \qquad 1/p + 1/p' = 1,$$

provided that $w \in A_p(dx)$. If $w \in A_1(dx)$, (9) is equivalent to

$$\operatorname*{ess\,sup}_I \{|f(x) - f_I| w(x)^{-1}\} \leqslant c\left[\int_I w(x)\right]^\varepsilon,$$

the ess sup being with respect to Lebesgue measure. Results of this type were proved in [2], using duality methods.

Taking μ and ν to satisfy (1), $F(I) = \nu(I)^\varepsilon$ and $g(x) \equiv 1$, one obtains that for any $p \geqslant 1$, the condition

$$\int_I |f(x) - f_I|^{p'} d\mu(x) \leqslant c\mu(I)\nu(I)^{\varepsilon p'}, \qquad f_I = f_{I,d\mu}$$

is equivalent to

$$|f(x) - f_I| \leqslant c\nu(I)^\varepsilon \quad \text{for a.e. } (d\mu)x \in I,$$

since in this case g trivially satisfies the A_p condition for any $p \geqslant 1$. This may be found in [4].

Proof of the Theorem. The hard part of the proof is the implication (6) $\Rightarrow$ (7), the opposite one being an immediate corollary of Hölder's inequality. The proof we give is based on the methods used in [1] and [4]. We would like to thank Professor C. Fefferman for a useful comment concerning the proof of Lemma 1.

LEMMA 1. *Let μ and ν satisfy* (1), *and F satisfy* (4) *and* (5). *If*

$$\int_I |f(x) - f_I| d\nu(x) \leqslant c\mu(I)F(I), \qquad f_I = f_{I,d\nu}, \tag{10}$$

then for a.e. $(d\nu)x \in I$,

$$|f(x) - f_I| \leqslant cF(I)\sup\{(\mu(J)/\nu(J)) \colon x \in J \subset 2I\}. \tag{11}$$

PROOF. For $x \in I$, let I_x denote the cube centered at x with the same edge length as I. We first claim that

$$|f_I - f_{I_x}| \leqslant cF(I)\frac{\mu(I)}{\nu(I)}. \tag{12}$$

In fact,

$$\begin{aligned} |f_I - f_{I_x}| &\leqslant |f_I - f_{2I}| + |f_{2I} - f_{I_x}| \\ &= \left|\frac{1}{\nu(I)}\int_I \{f(t) - f_{2I}\} d\nu(t)\right| + \left|\frac{1}{\nu(I_x)}\int_{I_x} \{f(t) - f_{2I}\} d\nu(t)\right|. \end{aligned}$$

Since both I and I_x lie in $2I$, and ν satisfies (1), each of the last two terms is majorized by

$$\frac{c}{\nu(I)}\int_{2I} |f(t) - f_{2I}| d\nu(t).$$

Using (10) to estimate the last integral, and then applying (1), (4) and (5), we obtain (12).

Next, we will show that for a.e. $(d\nu)x \in I$,

$$|f(x) - f_{I_x}| \leqslant cF(I)\sup\{(\mu(J)/\nu(J)) \colon x \in J \subset 2I\}. \tag{13}$$

To see this, set $I_k = 2^{-k}I_x$, and note that for a.e. $(d\nu)x \in I$, differentation of the integral gives

$$|f(x) - f_{I_x}| \leqslant \sum_{k=0}^{\infty} |f_{I_k} - f_{I_{k+1}}|.$$

However,

$$\begin{aligned} |f_{I_k} - f_{I_{k+1}}| &\leqslant \frac{1}{\nu(I_{k+1})}\int_{I_{k+1}} |f(t) - f_{I_k}| d\nu(t) \\ &\leqslant \frac{c}{\nu(I_k)}\int_{I_k} |f(t) - f_{I_k}| d\nu(t) \end{aligned}$$

by (1) and the fact that $I_{k+1} \subset I_k$. Hence, by (10),

$$|f_{I_k} - f_{I_{k+1}}| \leqslant \frac{c}{\nu(I_k)} \mu(I_k) F(I_k).$$

Since $2^k I_k = I_x$, (4) and (5) imply that $F(I_k) \leqslant c\alpha^{-k} F(I)$. Combining estimates, we obtain that for a.e. $(d\nu)x \in I$,

$$\begin{aligned} |f(x) - f_{I_x}| &\leqslant cF(I) \sum_{k=0}^{\infty} \alpha^{-k} \frac{\mu(I_k)}{\nu(I_k)} \\ &\leqslant cF(I) \left\{ \sup \frac{\mu(J)}{\nu(J)} \right\} \sum_{k=0}^{\infty} \alpha^{-k}, \end{aligned}$$

where the sup is taken over all cubes J such that $x \in J$ and $J \subset 2I$. Since $\alpha > 1$, the series converges and (13) follows. Combining (12) and (13) concludes the proof of the lemma.

We shall also use the following fact.

LEMMA 2. *Let ν be a Borel measure which satisfies* (1), *and let f_ν^* denote the Hardy-Littlewood maximal function of f with respect to ν*:

$$f_\nu^*(x) = \sup \left\{ \frac{1}{\nu(I)} \int_I |f(t)| \, d\nu(t) \colon x \in I \right\}.$$

If $u \in A_p(d\nu)$, $1 < p < \infty$, then there is a constant c, independent of f, such that

$$\int_{R^n} f_\nu^*(x)^p u(x) \, d\nu(x) \leqslant c \int_{R^n} |f(x)|^p u(x) \, d\nu(x).$$

This is a restatement of Theorem 7 of [6].

We now turn to the proof of the theorem, first showing that the assumptions on $g\,d\mu$ and g^{-1} imply that μ satisfies (1). Hölder's inequality and $g^{-1} \in A_p(g\,d\mu)$ imply

$$\left[\int_I g \, d\mu \right]^p \leqslant \mu(I) \left[\int_I g^{p/(p-1)} d\mu \right]^{p-1} \leqslant c \left[\int_I g \, d\mu \right]^p. \tag{14}$$

Since the same relation holds for $2I$,

$$\mu(2I) \left[\int_I g^{p/(p-1)} d\mu \right]^{p-1} \leqslant c \left[\int_{2I} g \, d\mu \right]^p \leqslant c \left[\int_I g \, d\mu \right]^p,$$

by (1) for $g\,d\mu$. Hence, by the first inequality (14), $\mu(2I) \leqslant c\mu(I)$.

If we set $d\nu = g\,d\mu$, then hypothesis (6) of the theorem is the same as (10) of Lemma 1. Hence, by Lemma 1, for a.e. $(d\nu)x \in I$,

$$|f(x) - f_I| \leqslant cF(I) \sup\{(\mu(J)/\nu(J)) \colon x \in J \subset 2I\},$$

where $f_I = f_{I,d\nu}$. If χ_{2I} denotes the characteristic function of $2I$, it follows from the formula $\mu(J) = \int_J g^{-1}\,d\nu$ that

$$(15) \qquad |f(x) - f_I| \leqslant cF(I)[g^{-1}\chi_{2I}]_\nu^*(x) \quad \text{for a.e. } (d\nu)x \in I.$$

For $1 < p < \infty$, multiply both sides of (15) by $g(x)$, raise the result to the power p', and then integrate over I with respect to $d\mu$, obtaining

$$(16) \qquad \int_I \{|f - f_I|g\}^{p'}\,d\mu \leqslant cF(I)^{p'} \int_{R^n} \{[g^{-1}\chi_{2I}]_\nu^*\}^{p'} g^{p'-1}\,d\nu.$$

It is easy to see that the assumption $g^{-1} \in A_p(d\nu)$ is equivalent to $g^{p'-1} \in A_{p'}(d\nu)$. Hence, Lemma 2 shows that the expression on the right in (16) is at most

$$cF(I)^{p'} \int_{2I} g^{-p'} g^{p'-1}\,d\nu = cF(I)^{p'}\mu(2I).$$

Since μ satisfies (1), we obtain

$$\int_I \{|f - f_I|g\}^{p'}\,d\mu \leqslant cF(I)^{p'}\mu(I),$$

as desired. If $p = 1$, the hypothesis $g^{-1} \in A_1(d\nu)$ gives

$$[g^{-1}]_\nu^* \leqslant cg^{-1} \quad \text{a.e.}$$

Combining this with (15) proves (8). Thus, the proof of the theorem is complete.

As a final remark, we note that the proof given above does not yield the results on bounded mean oscillation mentioned previously, since these correspond to $F(I) \equiv 1$, which fails to satisfy (5) (α must strictly exceed 1 there, in order that the series in Lemma 1 converge).

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Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903 (Current address of R. L. Wheeden)

Current address (E. P. Lotkowski): Prudential Property and Casualty Insurance Company, One Woodbridge Center, Woodbridge, New Jersey 07095