THE EQUIVALENCE OF
VARIOUS LIPSCHITZ CONDITIONS ON THE
WEIGHTED MEAN OSCILLATION
OF A FUNCTION

EDWARD P. LOTKOWSKI AND RICHARD L. WHEEDEN

ABSTRACT. The main result establishes the equivalence of various Lipschitz-like conditions on the weighted mean oscillation over cubes of a function. Previously known results are obtained as special cases.

Introduction. In [3], John and Nirenberg proved that if \( f \) is a function of bounded mean oscillation, i.e., if

\[
\int_I |f(x) - f_I| \, dx \leq c|I|,
\]

for all cubes \( I \subset \mathbb{R}^n \), with \( c \) independent of \( I \), then there exist positive constants \( c_1 \) and \( c_2 \) depending only on \( n \) such that

\[
|\{x \in I : |f(x) - f_I| > \alpha\}| \leq c_1 e^{-c_2 \alpha/\alpha} |I|
\]

for all \( I \) and \( \alpha > 0 \). In [7] Muckenhoupt and Wheeden derived an analogous result for functions of weighted bounded mean oscillation, by which we mean those \( f \) with

\[
\int_I |f(x) - f_I| \, dx \leq c \int_I w(x) \, dx
\]

for all \( I \), where \( w \) is nonnegative and satisfies appropriate conditions.

Meyers [5] and Campanato [1] showed independently that the condition

\[
\int_I |f(x) - f_I| \, dx \leq c |I|^{1+\varepsilon}
\]

for an \( \varepsilon \) satisfying \( 0 < \varepsilon \leq 1 \) is equivalent (after redefining \( f \) in a set of measure 0) to Lipschitz continuity of order \( \varepsilon \). Two different weighted versions of this result were then obtained by Cuerva [2] and Lotkowski [4]. Our main theorem generalizes and unifies these latter results, as we shall indicate below.

We first list some definitions. If \( \mu \) is a Borel measure which is positive and...
finite on cubes and \( f \) is locally integrable with respect to \( \mu \), then \( f_{I, \mu} \) (or simply \( f_I \)) will denote the average of \( f \) over \( I \) with respect to \( \mu \):

\[
f_I = f_{I, \mu} = \frac{1}{\mu(I)} \int_I f(x) \, d\mu(x).
\]

For \( a > 0 \), \( aI \) will denote the cube concentric with \( I \) whose edge length is \( a \) times that of \( I \). We shall be concerned only with measures which satisfy the doubling condition

\[
\mu(2I) \leq c \mu(I)
\]

for every \( I \), with \( c \) independent of \( I \). It is an easy consequence of (1) that two cubes of equal size whose centers are at a distance bounded by a fixed multiple of their edge length have equivalent \( \mu \) measure. For \( 1 < p < \infty \), a nonnegative function \( u \) is said to belong to \( A_p(\mu) \) if there is a constant \( c \) such that for all \( I \)

\[
\left( \frac{1}{\mu(I)} \int_I u(x) \, d\mu(x) \right) \left( \frac{1}{\mu(I)} \int_I u(x)^{-1/(p-1)} \, d\mu(x) \right)^{p-1} \leq c.
\]

Similarly, \( u \) is said to belong to \( A_1(\mu) \) if

\[
u^*_\mu(x) \leq cu(x) \quad \text{a.e. (} \mu \text{)}
\]

with \( c \) independent of \( x \), where \( u^*_\mu \) is the Hardy-Littlewood maximal function of \( u \) with respect to \( \mu \):

\[
u^*_\mu(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I u(t) \, d\mu(t) : x \in I \right\}.
\]

\( F(I) \) will denote a positive function of cubes for which there exist constants \( \alpha, \beta \) and \( c \) such that \( 1 < \alpha < \beta, c > 0 \) and

\[
F(I) \leq c F(J) \quad \text{if} \quad I \subset J,
\]

\[
\alpha F(I) \leq F(2I) \leq \beta F(I).
\]

It is crucial for what follows that \( \alpha \) strictly exceed 1. An example of such an \( F \) is

\[
F(I) = [\mu_1(I)]^{e_1} \cdots [\mu_m(I)]^{e_m}
\]

with \( \mu_j \) satisfying (1) and \( e_j > 0 \) for \( 1 \leq j \leq m \).

We shall often use the same letter \( c \) to denote different constants which are independent of \( x \) and \( I \). For a function \( g \) which is nonzero a.e., \( g^{-1} \) will denote the function \( 1/g \).

Our main result is given by the following theorem.

**Theorem.** Suppose that \( g \) is nonnegative and locally integrable with respect to
that the measure $g \, d\mu$ satisfies (1), and that $F$ satisfies (4) and (5). If $1 \leq p < \infty$ and $g^{-1} \in A_p(g \, d\mu)$, then the condition

$$\int_I |f(x) - f_I| g(x) \, d\mu(x) \leq c \mu(I) F(I), \quad f_I = f_{I,d\mu},$$

is equivalent to

$$\int_I \{|f(x) - f_I| g(x)|^{p'} \, d\mu(x) \leq c \mu(I) F(I)^{p'}, \quad 1/p + 1/p' = 1.$$  

In case $p = 1$, we have $p' = \infty$, and (7) should be interpreted in the $L^\infty$ sense:

$$\operatorname{ess sup}_I \{|f(x) - f_I| g(x)| \leq c F(I),$$

the ess sup being taken with respect to $d\mu$, or equivalently $g \, d\mu$.

Before proceeding with the proof, we list two important special cases. First, given $\varepsilon > 0$ and a weight function $w$, we choose

$$\mu(I) = \int_I w(x) \, dx, \quad F(I) = \left[ \int_I w(x) \, dx \right]^{\varepsilon}, \quad g(x) = w(x)^{-1},$$

obtaining that for $1 < p < \infty$, the condition

$$\int_I |f(x) - f_I| w(x)^{-1/2} \, dx \leq c \left[ \int_I w(x) \, dx \right]^{1+\varepsilon}, \quad f_I = f_{I,dx},$$

is equivalent to

$$\int_I |f(x) - f_I|^{p'} w(x)^{-1/(p-1)} \, dx \leq c \left[ \int_I w(x) \, dx \right]^{1+\varepsilon p'}, \quad 1/p + 1/p' = 1,$$

provided that $w \in A_p(dx)$. If $w \in A_1(dx)$, (9) is equivalent to

$$\operatorname{ess sup}_I \{|f(x) - f_I| w(x)^{-1} \leq c \left[ \int_I w(x) \right]^{\varepsilon},$$

the ess sup being with respect to Lebesgue measure. Results of this type were proved in [2], using duality methods.

Taking $\mu$ and $v$ to satisfy (1), $F(I) = v(I)\varepsilon$ and $g(x) = 1$, one obtains that for any $p \geq 1$, the condition

$$\int_I |f(x) - f_I|^{p'} d\mu(x) \leq c \mu(I) v(I)^{p'}, \quad f_I = f_{I,d\mu},$$

is equivalent to

$$|f(x) - f_I| \leq cv(I)^{\varepsilon} \quad \text{for a.e.} \ (d\mu) x \in I,$$

since in this case $g$ trivially satisfies the $A_p$ condition for any $p \geq 1$. This may be found in [4].
Proof of the Theorem. The hard part of the proof is the implication (6) ⇒ (7), the opposite one being an immediate corollary of Hölder's inequality. The proof we give is based on the methods used in [1] and [4]. We would like to thank Professor C. Fefferman for a useful comment concerning the proof of Lemma 1.

Lemma 1. Let μ and ν satisfy (1), and F satisfy (4) and (5). If

\[ \int_I |f(x) - f_I| d\nu(x) \leq c \mu(I) F(I), \quad f_I = f_{I, \nu}, \]

then for a.e. \((\nu) x \in I,\)

\[ |f(x) - f_I| \leq c F(I) \sup\{(\mu(J)/\nu(J)): x \in J \subset 2I\}. \]

Proof. For \(x \in I,\) let \(I_x\) denote the cube centered at \(x\) with the same edge length as \(I.\) We first claim that

\[ |f_I - f_{I_x}| \leq c F(I) \frac{\mu(I)}{\nu(I)}. \]

In fact,

\[ |f_I - f_{I_x}| \leq |f_I - f_{2I_x}| + |f_{2I_x} - f_{I_x}| \]
\[ = \left| \frac{1}{\nu(I)} \int_I (f(t) - f_{2I_x}) d\nu(t) \right| + \left| \frac{1}{\nu(I_x)} \int_{I_x} (f(t) - f_{2I_x}) d\nu(t) \right|. \]

Using (10) to estimate the last integral, and then applying (1), (4) and (5), we obtain (12).

Next, we will show that for a.e. \((\nu) x \in I,\)

\[ |f(x) - f_{I_x}| \leq c F(I) \sup\{(\mu(J)/\nu(J)): x \in J \subset 2I\}. \]

To see this, set \(I_k = 2^{-k} I_x,\) and note that for a.e. \((\nu) x \in I,\) differentiation of the integral gives

\[ |f(x) - f_{I_x}| \leq \sum_{k=0}^{\infty} |f_k - f_{k+1}|. \]

However,

\[ |f_k - f_{k+1}| \leq \frac{1}{\nu(I_{k+1})} \int_{I_{k+1}} |f(t) - f_k| d\nu(t) \]
\[ \leq \frac{1}{\nu(I_k)} \int_{I_k} |f(t) - f_k| d\nu(t) \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
by (1) and the fact that $I_{k+1} \subseteq I_k$. Hence, by (10),

$$|f_{I_k} - f_{I_{k+1}}| \leq \frac{c}{\nu(I_k)} \mu(I_k) F(I_k).$$

Since $2^k I_k = I_e$, (4) and (5) imply that $F(I_k) \leq c \alpha^{-k} F(I)$. Combining estimates, we obtain that for a.e. $(d\nu)x \in I$,

$$|f(x) - f_{I_k}| \leq c F(I) \sum_{k=0}^{\infty} \frac{\alpha^{-k} \mu(I_k)}{\nu(I_k)} \leq c F(I) \left( \sup_{J} \frac{\mu(J)}{\nu(J)} \right) \sum_{k=0}^{\infty} \alpha^{-k},$$

where the sup is taken over all cubes $J$ such that $x \in J$ and $J \subseteq 2I$. Since $\alpha > 1$, the series converges and (13) follows. Combining (12) and (13) concludes the proof of the lemma.

We shall also use the following fact:

**Lemma 2.** Let $\nu$ be a Borel measure which satisfies $(1)$, and let $f_\nu^*$ denote the Hardy-Littlewood maximal function of $f$ with respect to $\nu$:

$$f_\nu^*(x) = \sup \left\{ \frac{1}{\nu(J)} \int_J |f| \, d\nu(i) : x \in J \right\}.$$

If $u \in A_p(d\nu)$, $1 < p < \infty$, then there is a constant $c$, independent of $f$, such that

$$\int_{\mathbb{R}^n} f_\nu^*(x)^p u(x) \, d\nu(x) \leq c \int_{\mathbb{R}^n} |f(x)|^p u(x) \, d\nu(x).$$

This is a restatement of Theorem 7 of [6].

We now turn to the proof of the theorem, first showing that the assumptions on $g d\mu$ and $g^{-1}$ imply that $\mu$ satisfies (1). Hölder's inequality and $g^{-1} \in A_p(g d\mu)$ imply

$$\left[ \int_I g \, d\mu \right]^p \leq \mu(I) \left[ \int_I g^{p/(p-1)} \, d\mu \right]^{p-1} \leq c \left[ \int_I g \, d\mu \right]^p. \quad (14)$$

Since the same relation holds for $2I$,

$$\mu(2I) \left[ \int_{2I} g^{p/(p-1)} \, d\mu \right]^{p-1} \leq c \left[ \int_{2I} g \, d\mu \right]^p \leq c \left[ \int_I g \, d\mu \right]^p,$$

by (1) for $g d\mu$. Hence, by the first inequality $(14)$, $\mu(2I) \leq c \mu(I)$.

If we set $d\nu = g \, d\mu$, then hypothesis (6) of the theorem is the same as (10) of Lemma 1. Hence, by Lemma 1, for a.e. $(d\nu)x \in I$,

$$|f(x) - f_{I_k}| \leq c F(I) \sup_{J} (\mu(J)/\nu(J)). \quad x \in J \subseteq 2I,$$
where $f_I = f_{I, dv}$. If $\chi_{2I}$ denotes the characteristic function of $2I$, it follows from the formula
\[ \mu(I) = \int_I g^{-1} dv \]
that
\[ |f(x) - f_I| \leq cF(I)[g^{-1}\chi_{2I}]^*(x) \quad \text{for a.e. } (dv) x \in I. \tag{15} \]

For $1 < p < \infty$, multiply both sides of (15) by $g(x)$, raise the result to the power $p'$, and then integrate over $I$ with respect to $d\mu$, obtaining
\[ \int_I (|f - f_I| g)^{p'} d\mu \leq cF(I)^{p'} \int_{\mathbb{R}^n} \left( [g^{-1}\chi_{2I}]^* \right)^{p'} g^{p'-1} dv. \tag{16} \]

It is easy to see that the assumption $g^{-1} \in A_p(dv)$ is equivalent to $g^{p-1} \in A_p(dv)$. Hence, Lemma 2 shows that the expression on the right in (16) is at most
\[ cF(I)^{p'} \int_{2I} g^{-p'} g^{p'-1} dv = cF(I)^{p'} \mu(2I). \]

Since $\mu$ satisfies (1), we obtain
\[ \int_I (|f - f_I| g)^{p'} d\mu \leq cF(I)^{p'} \mu(I), \]
as desired. If $p = 1$, the hypothesis $g^{-1} \in A_1(dv)$ gives
\[ [g^{-1}]^* \leq cg^{-1} \quad \text{a.e.} \]

Combining this with (15) proves (8). Thus, the proof of the theorem is complete.

As a final remark, we note that the proof given above does not yield the results on bounded mean oscillation mentioned previously, since these correspond to $F(I) = 1$, which fails to satisfy (5) ($\alpha$ must strictly exceed 1 there, in order that the series in Lemma 1 converge).

**References**