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A GENERALIZATION OF THE ARITHMETIC-GEOMETRIC MEANS INEQUALITY[1]

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ABSTRACT. It is shown that the arithmetic mean of $x_1 w_1, \ldots, x_n w_n$ exceeds the geometric mean of $x_1, \ldots, x_n$ unless all the x's are equal, where $w_1, \ldots, w_n$ depend on $x_1, \ldots, x_n$ and satisfy $0 \leqslant w_i < 1$ unless $x_i = \min x_k$. This inequality is then applied to an integral inequality for functions y defined on $[0, \infty)$ with $y^{(k)}$ convex and 0 at 0 for $0 \leqslant k < n$.

The inequality between the arithmetic and geometric means of n positive numbers x_i may be written

$$\text{(A-G)} \qquad \sum_{i=1}^{n} x_i^n \geqslant n \prod_{i=1}^{n} x_i .$$

Our generalization of this inequality gives a simple formula for factors w_i, $0 \leqslant w_i \leqslant 1$, which allow us to replace x_i^n in (A-G) with $w_i x_i^n$ and maintain the inequality, with $w_i < 1$ unless x_i is minimal among the x's. This is carried out in §1. In §2 we apply this to an integral inequality. In §3 we turn to the question of finding factors w_i which give an identity.

1. **A generalization of the theorem of the arithmetic and geometric means.** We begin with formulas we need to specify the factors w_i mentioned before. Define functions q_n for $t \geqslant 0$ by

$$(1.1) \qquad q_n(t) = 1 - (1-t)^n \quad \text{for } 0 \leqslant t \leqslant 1; \qquad q_n(t) \equiv 1 \quad \text{for } t > 1.$$

For $n > 1$, q_n is C^1 and $q_n'(t) \equiv 0$ for $t \geqslant 1$. Next define $Q_n(r)$ for $r = (r_1, \ldots, r_n)$ (all $r_i > 0$) by $Q_n(r) = q_n(r_1) \cdots q_n(r_n)$. It is clear that $Q_n(r) < 1$ unless $r_i \geqslant 1$ for all i.

THEOREM 1. *If $x_1, \ldots, x_n$ are positive numbers then, with $x = (x_1, \ldots, x_n)$,*

$$(1.2) \qquad \sum_{i=1}^{n} Q_n\left(\frac{x}{x_i}\right) x_i^n \geqslant n \prod_{i=1}^{n} x_i .$$

There is equality if and only if all the x_i have the same value.

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In passing we note that $Q_n(x/x_i)$ involves only $n-1$ factors, that (1.2) is an identity when $n=2$, and that $\{Q_n(x/x_i)\}$ and $\{x_i\}$ are oppositely ordered.

To prove Theorem 1 we apply the method of Lagrange multipliers to the function $F(x)$ defined by the left-hand side of (1.2), subject to the constraint $x_1 x_2 \cdots x_n = 1$. Since F is homogeneous of degree n, we use Euler's formula and the constraint to arrive at the equations $x_i \partial F/\partial x_i = \lambda = F(x)$ at critical points (x, λ) of the function $\Phi(x,\lambda) = F(x) + \lambda(1-(x_1 \cdots x_n))$. We may suppose without loss that x_1 is among the largest of the x_i. Then

$$x_1 \frac{\partial F}{\partial x_1} = n x_1^n Q_n\left(\frac{x}{x_1}\right) + x_1 \sum_{k=2}^{n} x_1^n Q_n\left(\frac{x}{x_1}\right) \frac{q_n'(x_k/x_1)}{q_n(x_k/x_1)} \cdot \left(-\frac{x_k}{x_1^2}\right)$$
$$+ x_1 \sum_{i=2}^{n} x_i^n \frac{\partial}{\partial x_1}\left(Q_n\left(\frac{x}{x_i}\right)\right).$$

Now

$$\frac{\partial}{\partial x_1}\left(Q_n\left(\frac{x}{x_i}\right)\right) = q_n'\left(\frac{x_1}{x_i}\right)\frac{1}{x_i}\prod_{k=2}^{n} q_n\left(\frac{x_k}{x_i}\right) = 0 \quad \text{for } i > 1$$

since $x_1 \geqslant x_i$. Thus we have

$$x_1 \frac{\partial F}{\partial x_1} = n x_1^n Q_n\left(\frac{x}{x_1}\right)\left\{1 - \sum_{k=2}^{n} \frac{(1 - x_k/x_1)^{n-1}}{q_n(x_k/x_1)} \cdot \frac{x_k}{x_1}\right\} = \lambda. \tag{1.3}$$

It is convenient to define functions $p_n(t)$ by

$$t p_n(t) = q_{n+1}(t), \tag{1.4}$$

so that $p_n(t) = \sum_{r=0}^{n} (1-t)^r$ for $0 \leqslant t \leqslant 1$ and $p_n(t) = 1/t$ for $t > 1$.

REMARK. We shall not need the $p_n(t)$ for any values of $t > 1$, but we note that when the factors $Q_n(x/x_i)$ in (1.1) are expressed in terms of $p_{n-1}(x_j/x_i)$, and $n x_1 \cdots x_n$ is cancelled from both sides, that inequality assumes the form

$$\frac{1}{n} \sum_{i=1}^{n} \left\{ \prod_{k=1}^{n} p_{n-1}\left(\frac{x_k}{x_i}\right) \right\} \geqslant 1. \tag{1.5}$$

When we use the function p_{n-1} and the constraint $x_1 \cdots x_n = 1$, (1.3) becomes

$$n \cdot \left(\prod_{i=2}^{n} p_{n-1}\left(\frac{x_i}{x_1}\right)\right)\left\{1 - \sum_{k=2}^{n} \frac{(1 - x_k/x_1)^{n-1}}{p_{n-1}(x_k/x_1)}\right\} = \lambda. \tag{1.6}$$

Since we also have $F(x) = \lambda$ at a critical point, Theorem 1 is reduced to the following lemma, in which n has been replaced by $n+1$.

LEMMA 1. *If* $0 \leqslant t_k \leqslant 1$, $1 \leqslant k \leqslant n$, *then*

$$\left(\prod_{i=1}^{n} p_n(t_i)\right)\left\{1 - \sum_{k=1}^{n} \frac{(1-t_k)^n}{p_n(t_k)}\right\} \geqslant 1. \tag{1.7}$$

Equality holds for $n > 1$ only when all the t_k are equal to 1.

The inequality holds when all the t_k are zero, for then the left-hand side of (1.7) is $(n+1)^n\{1 - n/(n+1)\} = (n+1)^{n-1} > 1$ (unless $n = 1$; in that case (1.7) is an identity).

We shall prove Lemma 1 by showing that when all but one of the variables t_k are held fixed, the left-hand side of (1.7) decreases strictly in the free variable. By symmetry we may replace t_n by t and estimate the derivative of

$$f(t) \equiv p_n(t)\left\{1 - \sum_{k=1}^{n-1} \frac{(1-t_k)^n}{p_n(t_k)} - \frac{(1-t)^n}{p_n(t)}\right\}, \qquad 0 \leqslant t \leqslant 1. \tag{1.8}$$

For brevity let s denote the summation within the curly brackets. Then $f(t) = p_n(t)(1-s) - (1-t)^n$, so

$$f'(t) = p_n'(t)(1-s) + n(1-t)^{n-1} = p_{n-1}'(t) - sp_n'(t). \tag{1.9}$$

We now need an estimate for s, which is a sum of terms $g(u) = (1-u)^n/p_n(u)$. Let us verify that $g'(u) < 0$ for $0 \leqslant u < 1$:

$$g'(u) = (1-u)^{n-1}\{-np_n(u) - (1-u)p_n'(u)\}p_n(u)^{-2};$$

the expression in curly brackets is $-n - \sum_1^n (n-r)(1-u)^r < 0$. It follows that

$$s = \sum_{k=1}^{n-1} \frac{(1-t_k)^n}{p_n(t_k)} < \frac{n-1}{n+1}$$

unless all the t_k are 0, $1 \leqslant k < n$. With this estimate for s, and the fact that $p_n'(t) < 0$, (1.9) becomes

$$f'(t) < p_{n-1}'(t) - \frac{n-1}{n+1}p_n'(t), \quad \text{unless all } t_k = 0,\ 1 \leqslant k < n. \tag{1.10}$$

Now

$$\frac{-p_n'(t)}{-p_{n-1}'(t)} = 1 + \frac{n(1-t)^{n-1}}{\sum_1^{n-1} r(1-t)^{r-1}} = 1 + \frac{n}{\sum_1^{n-1} r(1-t)^{r-n}};$$

thus the ratio has its largest value when $t = 0$. Accordingly,

$$-p_n'(t) < \frac{n+1}{n-1}(-p_{n-1}'(t)) \tag{1.11}$$

unless $t = 0$. From (1.10), (1.11) we have $f'(t) < 0$ unless $t = 0$ and all t_k are zero. It follows that on the cube $\{0 \leqslant t_i \leqslant 1, i = 1, \ldots, n\}$, the function

defined by the expression on the left-hand side of (1.7) decreases strictly in each variable from a value of $(n+1)^{n-1}$ at the origin, to the value 1 when all the t_k are 1. This completes the proof of Lemma 1, and Theorem 1 follows.

We mention two questions. (1) Is there an ' elementary ' proof of Theorem 1, as defined in §1.7 of *Inequalities* [3]? Is there an argument which itself uses (A-G)? (2) Can one prove (1.5) directly by using the following property of the functions $p_n(t)$ (see (1.4)):

$$p_n(t) - (1/n)tp_n'(t) = ((n+1)/n)p_{n-1}(t)\ ?$$

2. **An application of Theorem 1: two integral inequalities.** The first inequality we give below arose from the question of continuability of solutions of $x^{(n)}(t) + a(t)f(x(t)) = 0$ across intervals in which $a(t)$ is negative (here $xf(x) > 0$ for $x \neq 0$). For more on this see Burton [1] and Fink [2].

THEOREM 2. *Suppose* $y \in C^{n+1}([0,\infty))$, $y^{(k)}(0) = 0$ *for* $0 \leqslant k < n$, $y^{(n)}(0) \geqslant 0$ *and* $y^{(n+1)}(x) \geqslant 0$ *for* $x \geqslant 0$. *Then*

$$\begin{aligned} y(x)^{n+1} \leqslant (n+1)^n \int_0^x y(u_1) \int_0^{u_1} y(u_2) \int_0^{u_2} \\ \cdots \int_0^{u_{n-2}} y(u_{n-1}) \int_0^{u_{n-1}} y(u_n) y^{(n)}(u_n)\, du_n \cdots du_1, \end{aligned} \tag{2.1}$$

with equality if and only if $y(x) = cx^n$. *The condition* $y^{(n+1)}(x) \geqslant 0$ *cannot be dropped.*

This result is an immediate consequence of

THEOREM 2′. *Let* μ *be a nonnegative measure on* $[0,\infty)$. *Set*

$$y(x) = \int_0^x (x-t)^n/n!\, d\mu(t).$$

Then (2.1) *holds, with equality if and only if*

$$y(x) = \begin{cases} 0, & 0 \leqslant x \leqslant x_0, \\ c(x-x_0)^n, & x > x_0, \end{cases}$$

where $c \geqslant 0$, $x_0 \geqslant 0$.

To prove Theorem 2′ we express the quantities on each side of (2.1) as repeated integrals of $d\mu$, and simplify. The kernel for the right-hand side can be averaged over permutations, giving an expression to which Theorem 1 may be applied. To carry this out let us set $S(x,u) = S(x, u_1, \ldots, u_n) = 1$ if $x > u_1 > \cdots > u_n > 0$, and set $S(x,u) = 0$ otherwise. We also write t_+^k for $\chi_+(t)t^k$, where χ_+ is the characteristic (or indicator) function of the positive real numbers. Now the integral on the right in (2.1) becomes

$$\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n+1} \left\{ \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} S(x,u) \prod_{k=1}^n \frac{(u_k - t_k)_+^n}{n!} \chi_+(u_n - t_{n+1})\, du \right\} d\mu(t_1) \cdots d\mu(t_{n+1}).$$

If we now average the quantity in curly brackets over all permutations of $(t_1, \ldots, t_{n+1})$ we obtain

$$K(x,t) \equiv \frac{1}{(n+1)!}\sum_{j=1}^{n+1}\sum_P \int S(x,u)\prod_{k=1}^{n}\frac{(u_k-(P\pi_j t)_k)_+^n}{n!}\chi_+(u_n-t_j)\,du,$$

where (i) the integral is over $\{u \in \mathbf{R}^n\colon u_k \geqslant 0, 1 \leqslant k \leqslant n\}$, (ii) $\pi_j t$ is the vector in $\mathbf{R}^n$ obtained by deleting t_j from $(t_1, \ldots, t_{n+1})$, (iii) P is an $n \times n$ permutation matrix. The sum is over all such P. Let us write $F(u) = \prod_{k=1}^n (u_k)_+^n/n!$, and then for each P replace u by Pu in the integral:

$$K(x,t) = \frac{1}{(n+1)!}\sum_{j=1}^{n+1}\int\left\{\sum_P S(x,Pu)\chi_+((Pu)_n - t_j)\right\}F(u-\pi_j t)\,du$$

since $F(Pu) = F(u)$. The sum in curly brackets (for almost all u) consists of only one nonzero term, for $S(x,Pu) = 0$ unless the coordinates of Pu are in decreasing order, are all different, and are all less than x. Then $(Pu)_n = \min_k u_k$, so

$$K(x,t) = \frac{1}{(n+1)!}\sum_{j=1}^{n+1}\int_{t_j}^{x}\cdots\int_{t_j}^{x} F(u-\pi_j t)\,du_1\cdots du_n$$

(2.2)

$$= \frac{1}{(n+1)!}\sum_{j=1}^{n+1}\prod_{k=1;k\neq j}^{n+1}\left\{\frac{(x-t_k)_+^{n+1}}{(n+1)!} - \frac{(t_j-t_k)_+^{n+1}}{(n+1)!}\right\}.$$

Now $y(x)^{n+1}$ has kernel $\prod_{j=1}^{n+1}(x-t_j)_+^n/n!$, so we have to show that when $0 \leqslant t_j \leqslant x$ for all j,

$$(n+1)^n K(x,t) - \prod_{j=1}^{n+1}\frac{(x-t_j)_+^n}{n!} \geqslant 0, \tag{2.3}$$

with equality if and only if $t_1 = \cdots = t_{n+1}$. This is immediate if one of the $t_j = x$. Write $v_k = x - t_k$, so $(t_j - t_k)_+ = (v_k - v_j)_+$. Then when appropriate substitutions and cancellations of factorials are made, the left side of (2.3) becomes (when $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_{n+1} > 0$):

$$\sum_{j=1}^{n+1}\left\{\prod_{k<j} v_k^{n+1}\right\}\left\{\prod_{j<k\leqslant n+1}(v_k^{n+1} - (v_k - v_j)^{n+1})\right\} - (n+1)\prod_1^{n+1} v_k^n$$

$$= \sum_{j=1}^{n+1}\left\{\prod_{k<j} v_k^{n+1}\right\}\left\{\prod_{j<k\leqslant n+1} v_k^{n+1} q_{n+1}\left(\frac{v_j}{v_k}\right)\right\} - (n+1)\prod_1^{n+1} v_k^n$$

$$= \left(\prod_{k=1}^{n+1} v_k^{n+1}\right)\sum_{j=1}^{n+1} v_j^{-n-1}\left\{\prod_{j<k\leqslant n+1} q_{n+1}\left(\frac{v_j}{v_k}\right)\right\} - (n+1)\prod_1^{n+1} v_k^n.$$

Finally, we cancel $\prod_{k=1}^{n+1} v_k^{n+1}$, use the definition of Q_{n+1} preceding Theorem 1, set $x_k = 1/v_k$, and obtain (1.2) (for $n+1$ instead of n). Thus by Theorem 1,

$$0 < \int_0^\infty \cdots \int_0^\infty \left\{ (n+1)^n K(x,t) - \prod_{j=1}^{n+1} \frac{(x-t_j)_+^n}{n!} \right\} d\mu(t_1) \cdots d\mu(t_{n+1})$$

unless $d\mu(t_1) \cdots d\mu(t_{n+1})$ is carried by the diagonal. Hence $d\mu = c\delta_{x_0}$ in this case, with $c > 0$. This completes the proof of Theorem 2′.

When we use $K(x, t)$, as given by the integral in (2.2), and $n + 1$ nonnegative measures $\mu_1, \ldots, \mu_{n+1}$ on $[0, \infty)$ we obtain the following inequality.

THEOREM 3. *Let $\mu_1, \ldots, \mu_{n+1}$ be nonnegative measures on $[0, \infty)$. Set $y_j(x) = \int_0^x (x-t)_+^n/n!\, d\mu_j(t)$, and let $\wedge u = \min_{1 \leqslant k \leqslant n} u_k$. Then*

$$y_1(x) \cdots y_{n+1}(x) \leqslant \frac{(n+1)^{n-1}}{n!} \sum_{j=1}^{n+1} \underbrace{\int_0^x \cdots \int_0^x}_{n} y_1(u_1) \cdots y_j^{(n)}(\wedge u) \cdots y_{n+1}(u_n)\, du.$$

Equality holds if and only if $n = 1$ or $(n > 1)$ $y_k(x) = c_k(x - x_0)_+^n$, $c_k \geqslant 0$, $x_0 \geqslant 0$. (The product under the integral signs contains $y_k(u_k)$ for $k < j$, $y_{k+1}(u_k)$ for $k > j$ and $y_j^{(n)}(\wedge u)$.)

Thus when all the μ_k's are the same we get Theorem 2′ in the form

$$y(x)^{n+1} \leqslant \frac{(n+1)^n}{n!} \underbrace{\int_0^x \cdots \int_0^x}_{n} y(u_1) \cdots y(u_n) y^{(n)}(\wedge u)\, du.$$

In some sense this inequality thus depends "as little as possible" on the behavior of $y^{(n)}(t)$ near $t = x$.

3. **Additional remarks.** One approach to proving an inequality involves starting with an identity from which the inequality follows immediately. Remarks (3.1)–(3.3) point in this direction. Remarks (3.4), (3.5) give other ways of generalizing (A-G).

(3.1) The inequality $xy \leqslant (1/p)x^p + (1/q)y^q$ $(x \geqslant 0, y \geqslant 0, 1/p + 1/q = 1, p > 1, q > 1)$ can be replaced by the formula

$$xy = (1/p)x^p \omega_p(y/x^{p-1}) + (1/q)y^q \omega_q(x/y^{q-1}),$$

where

$$\omega_p(t) = \begin{cases} pt(1 - (1/q)t^{q-1}), & 0 \leqslant t \leqslant 1, \\ 1, & t > 1, \end{cases}$$

and $\omega_q(t)$ is defined by interchanging p and q. It is easy to check that ω_r is an increasing concave function, and that $\omega_r(t) < 1$ for $0 \leqslant t < 1$.

(3.2) It follows that we can arrive at a formula

$$x_1 \cdots x_n = \sum_{i=1}^n \frac{x_i^{P_i}}{P_i} \omega_i(x) \qquad \left(P_i > 1, \frac{1}{P_1} + \cdots + \frac{1}{P_n} = 1 \right)$$

where $0 \leqslant \omega_i(x) \leqslant 1$, and $\omega_i(x) < 1$ unless $x_i = \min_k x_k$. There are many ways to do this, so an averaging is indicated.

(3.3) Similar things can be done, starting with Young's inequality $xy \leqslant A(x) + \bar{A}(y)$ for complementary convex functions [4, §2.7].

(3.4) Another inequality, involving ordering of the x_i in (A-G) is: If $x_1 \geqslant \cdots \geqslant x_n \geqslant 0$ then

$$\sum_{i=1}^{n} x_i^n - n \prod_{i=1}^{n} x_i \geqslant (x_1 - x_n)^n.$$

This is proved by ' separating out' x_n.

(3.5) There exists a number $Q(x) > 1$ (unless all x_i are equal) depending on $x_1, \ldots, x_n$ such that

$$\sum_{i=1}^{n} x_i^n > Q(x) \cdot n \prod_{i=1}^{n} x_i.$$

This is proved by writing x in place of x_n, then minimizing $(b + x^n)/x$; $Q(x)$ can be taken to be $\rho \equiv (b_j/(n-1))^{1-1/n} \cdot 1/a_j$, where $b_j = \sum_{i=1;i\neq j}^{n} x_i^n$, and $a_j = \prod_{i=1;i\neq j}^{n} x_i$, with j chosen to maximize this expression. It exceeds unity because when we set $y_k^{n-1} = x_k^n$,

$$\rho = \left(\frac{1}{n-1} \sum_{k\neq j} y_k^{n-1} \Big/ \prod_{k\neq j} y_k\right)^{1-1/n}$$

This process can be continued step by step until we arrive at a value for ρ of $\{\frac{1}{2}((\max x_k)^n + (\min x_k)^n)/(\max x_k)^{n/2} \cdot (\min x_k)^{n/2}\}^{2/n}$ (at each step we eliminated the value of x_j^n closest to the average).

This inequality has the form of one due to C. L. Siegel [5] (see [4, Theorem 4, p. 85]).

(3.6) As is the case with (A-G), Theorem 1 does not give Carlemann's inequality directly [3, Theorem 334].

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