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MEROMORPHIC FUNCTIONS AND SMOOTH ANALYTIC FUNCTIONS

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ABSTRACT. Meromorphic functions with many zeroes can have logarithmic derivatives that are relatively smooth. We prove this, with a new construction of smooth analytic functions with many zeroes. Our examples belong to the theory of differential fields of functions.

In this note we consider functions y, meromorphic in the disk $|z| < 1$, and their logarithmic derivatives $L(y) = y'/y$. Plainly, zeroes and poles of y are poles of $L(y)$, but the multiplicity of the zeroes and poles is not easily controlled. When $L(y)$ is a function of bounded characteristic, i.e. a quotient of bounded analytic functions in $|z| < 1$, the sequence $S = (z_k)_1^\infty$ of zeroes of y must fulfill the Blaschke condition $\Sigma 1 - |z_k| < +\infty$, but S. Bank proved recently [1] that the multiplicities of the zeroes z_k can be determined arbitrarily, if only $\Sigma 1 - |z_k| < +\infty$.

The set of functions of bounded characteristic forms a field, but not a differential field; indeed, W. Rudin [4] constructed a bounded analytic function g such that $\int_0^1 |g'(re^{i\theta})|\, dr = +\infty$ for almost all θ, whence g' is not of bounded characteristic. (See also [5].) Let now A^∞ be the class of functions g, such that each derivative $g^{(n)}$ is bounded in $|z| < 1$, and M^∞ the field of quotients of A^∞. Clearly M^∞ is a differential field of functions.

THEOREM. *Let $g \in A^\infty$, $g \not\equiv 0$, and let $S = (z_k)_1^\infty$ be the zero-set of g in $|z| < 1$. Then for any sequence (n_k) of nonnegative integers, there is a meromorphic function y, with zeroes at z_k of multiplicity n_k (and no other zeroes) such that $L(y)$ is in M^∞.*

In the proof of our theorem we need a precise description of possible zero-sets S, obtained in [3] and [7]. In an Appendix we derive this description by a method rather different from [3], [7].

It is easy to derive a necessary property of S in terms of the function $\rho(z) = \inf\{|z - s|: s \in S\}$, since $|g(e^{i\theta})| \leqslant C\rho(e^{i\theta})$. Now it is clear that $\log \rho(e^{i\theta})$ must be integrable on $(0, 2\pi)$; in combination with the Blaschke condition, this is sufficient.

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Let now $S = (z_k)$ be a zero-set as in the statement of the theorem. We choose $z_k^* \not\in S$ so that

$$2|z_k^* - z_k| < 1 - |z_k| \quad \text{and} \quad n_k|z_k - z_k^*| \leqslant C_r(1 - |z_k|)^{r+2}k^{-2}$$

for $r = 1, 2, 3, \ldots$ We set $S_1 = (z_k) \cup (z_k^*)$, and observe S_1 is a Blaschke sequence, while $|e^{i\theta} - z_k| \leqslant 2|e^{i\theta} - z_k^*|$, so thatat S_1 is the zero-set of some function g_1 in A^∞.

We assert now that the series

$$h_N = g_1 \sum_1^N n_k\left[(z - z_k)^{-1} - (z - z_k^*)^{-1}\right]$$

converges uniformly on $|z| < 1$, together with all of its derivatives. To verify this, we have only to estimate the derivatives on the boundary $|z| = 1$, and by Leibniz' formula we can omit the factor g_1. The rth derivative is then

$$\sum_1^N (-1)^r r!\, n_k\left[(z - z_k)^{-1-r} - (z - z_k^*)^{-1-r}\right].$$

Now

$$(\partial/\partial w)(z - w)^{-1-r} = (1 + r)(z - w)^{-2-r},$$

so that

$$\left|(z - z_k)^{-1-r} - (z - z_k^*)^{-1-r}\right| \leqslant C_r'|z_k - z_k^*|(1 - |z_k|)^{-2-r}$$

on the boundary $|z| = 1$. We find that $\lim h_N = h$ belongs to A^∞, and for the function $y = \prod(z - z_k)^{n_k}(z - z_k^*)^{-n_k}$ we have $L(y) = hg_1^{-1}$in M^∞.

Appendix. Let S be a Blaschke sequence such that $\log \rho(e^{i\theta})$ is integrable. There is a function $\delta(e^{i\theta})$, of class C^2 on $|z| = 1$, such that $C_1\rho^2(e^{i\theta}) \leqslant \delta(e^{i\theta}) \leqslant \rho^2(e^{i\theta})$. In fact δ is nothing but the square of the "regularized distance" Δ to the set S [6, p. 171]. Let now D^+ be the region bounded by the curve $r = 1 + \delta(e^{i\theta})$, so $D^+ \supseteq D$ and D^+ is a region of class C^2. There exists a conformal mapping Φ of D^+ onto D such that Φ and Φ^{-1} both have derivatives continuous up to the boundary, and even Hölder-continuous [2, p.374]; whence $a|z_1 - z_2| \leqslant |\Phi(z_1) - \Phi(z_2)| \leqslant b|z_1 - z_2|$ for certain constants $a > 0$, $b > 0$ (Kellogg's theorem). The distance of z_k from ∂D^+ is at most $1 - |z_k| + \delta(e^{i\theta})$ if $z_k = e^{i\theta}|z_k|$, so the distance is at most $2(1 - |z_k|)$. Consequently $1 - |\Phi(z_k)| = O(1 - |z_k|)$, whence $\Phi(S)$ is a Blaschke sequence in D, and there is a bounded analytic function B, on D^+, with zero-set S.

Let μ_z be the harmonic measure on ∂D^+ for the point z. By the differentiability properties of Φ and Φ^{-1}, we see that $L^1(d\theta)$ and $L^1(d\mu_0)$ can be identified. Moreover, elementary geometry yields the inequality $\rho(Re^{i\theta}) > \rho(e^{i\theta})$ when $R > 1$, so that $\log \rho(w)$ belongs to $L^1(d\mu_0)$. By a classical method we can find a monotone function $\psi(t)$ on $t > 0$, such that $\psi(t) \geqslant t + 1$ and $\psi(t)/t \to +\infty$ as $t \to +\infty$ and $\psi(|\log \rho(w)|)$ is in $L^1(d\mu_0)$. Let u be its Poisson integral on D^+, v the harmonic conjugate of u, and $g = e^{-u-iv}B$.

Before completing the proof that g belongs to A^∞, we observe an inequality on μ_z for z near ∂D^+. Let Γ_z be the part of the boundary defined by the inequality $|z - w| < 3d(z, \partial D^+)$. Using the continuity of Φ' we find that $\mu_z(\Gamma_z) \geqslant a > 0$ for all z in D^+. On Γ_z we have $\rho(w) \leqslant \rho(z) + 3d(z, \partial D^+)$, so that, if $\rho(z) + d(z, \partial D^+)$ is small, $u(z)$ is a large multiple of

$$-\log\left[\rho(z) + d(z, \partial D^+)\right],$$

and $|g(z)|$ is bounded by a large power of $\rho + d$. Thus, for each $N > 0$

$$|g(z)| \leqslant C_N\left[\rho(z) + d(z, \partial D^+)\right]^N.$$

Around each z in D^+ we draw a disk of radius $d(z, \partial D^+)/2$, and observe that the bound for g is increased by at most 2^N. Cauchy's formulas give, for $r = 0, 1, 2, \ldots,$

$$\left|g^{(r)}(z)\right| \leqslant C_{N,r}\left[\rho(z) + d(z, \partial D^+)\right]^N / d^r(z, \partial D).$$

Let $\varepsilon > 0$, and observe that on the set defined by the inequality $d(z, \partial D^+) \geqslant \varepsilon\rho^2(z)$, each $g^{(r)}$ is uniformly bounded. But for small $\varepsilon > 0$, this set contains the disk $|z| < 1$. Indeed $d(|z|e^{i\theta}, \partial D^+) \geqslant 1 - |z| + \varepsilon'\rho^2(e^{i\theta})$, so the inequality is true as soon as $\varepsilon'(1 - 2\varepsilon)^2 \geqslant \varepsilon$.

In the proof just completed, S could have contained points on ∂D, in which case all the derivatives $g^{(r)}$ vanish on $S \cap \partial D$. It is also worth remarking that a Blaschke sequence contained in a ball $|z - r| \leqslant 1 - r$ $(0 < r < 1)$ is a zero-set for A^∞, because $\rho(e^{i\theta}) \geqslant |e^{i\theta} - r| - 1 + r > a\theta^2$ for small θ.

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