MEROMORPHIC FUNCTIONS
AND SMOOTH ANALYTIC FUNCTIONS

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ABSTRACT. Meromorphic functions with many zeroes can have logarithmic derivatives that are relatively smooth. We prove this, with a new construction of smooth analytic functions with many zeroes. Our examples belong to the theory of differential fields of functions.

In this note we consider functions \( y \), meromorphic in the disk \( |z| < 1 \), and their logarithmic derivatives \( L(y) = y'/y \). Plainly, zeroes and poles of \( y \) are poles of \( L(y) \), but the multiplicity of the zeroes and poles is not easily controlled. When \( L(y) \) is a function of bounded characteristic, i.e. a quotient of bounded analytic functions in \( |z| < 1 \), the sequence \( S = (z_k) \) of zeroes of \( y \) must fulfill the Blaschke condition \( \sum |z_k| < +\infty \), but S. Bank proved recently [1] that the multiplicities of the zeroes \( z_k \) can be determined arbitrarily, if only \( \sum |z_k| < +\infty \).

The set of functions of bounded characteristic forms a field, but not a differential field; indeed, W. Rudin [4] constructed a bounded analytic function \( g \) such that \( \int |g(re^{i\theta})| \, dr = +\infty \) for almost all \( \theta \), whence \( g' \) is not of bounded characteristic. (See also [5].) Let now \( A^\infty \) be the class of functions \( g \), such that each derivative \( g^{(n)} \) is bounded in \( |z| < 1 \), and \( M^\infty \) the field of quotients of \( A^\infty \). Clearly \( M^\infty \) is a differential field of functions.

**Theorem.** Let \( g \in A^\infty \), \( g \neq 0 \), and let \( S = (z_k) \) be the zero-set of \( g \) in \( |z| < 1 \). Then for any sequence \( (n_k) \) of nonnegative integers, there is a meromorphic function \( y \), with zeroes at \( z_k \) of multiplicity \( n_k \) (and no other zeroes) such that \( L(y) \) is in \( M^\infty \).

In the proof of our theorem we need a precise description of possible zero-sets \( S \), obtained in [3] and [7]. In an Appendix we derive this description by a method rather different from [3], [7].

It is easy to derive a necessary property of \( S \) in terms of the function \( \rho(z) = \inf \{|z - s| : s \in S\} \), since \( |g(e^{i\theta})| \leq C \rho(e^{i\theta}) \). Now it is clear that \( \log \rho(e^{i\theta}) \) must be integrable on \( (0, 2\pi) \); in combination with the Blaschke condition, this is sufficient.
Let now $S = (z_k)$ be a zero-set as in the statement of the theorem. We choose $z_k^* \not\in S$ so that
\[ 2|z_k^* - z_k| < 1 - |z_k| \quad \text{and} \quad n_k |z_k - z_k^*| < C_r (1 - |z_k|)^{r+2k-2} \]
for $r = 1, 2, 3, \ldots$ We set $S_1 = (z_k) \cup (z_k^*)$, and observe $S_1$ is a Blaschke sequence, while $|e^{i\theta} - z_k| < 2|e^{i\theta} - z_k^*|$, so that $S_1$ is the zero-set of some function $g_1$ in $A^\infty$.

We assert now that the series
\[ h_N = g_1 \sum_{k=1}^{N} n_k [(z - z_k)^{-1} - (z - z_k^*)^{-1}] \]
converges uniformly on $|z| < 1$, together with all of its derivatives. To verify this, we have only to estimate the derivatives on the boundary $|z| = 1$, and by Leibniz' formula we can omit the factor $g_1$. The $r$th derivative is then
\[ \sum_{k=1}^{N} (-1)^r r! n_k [(z - z_k)^{-1-r} - (z - z_k^*)^{-1-r}] \]
Now
\[ (\partial / \partial w)(z - w)^{-1-r} = (1 + r)(z - w)^{-2-r}, \]
so that
\[ |(z - z_k)^{-1-r} - (z - z_k^*)^{-1-r}| \leq C_r |z_k - z_k^*|(1 - |z_k|)^{-2-r} \]
on the boundary $|z| = 1$. We find that $\lim h_N = h$ belongs to $A^\infty$, and for the function $y = \Pi(z - z_k)^{n_k}(z - z_k^*)^{-n_k}$ we have $L(y) = h g_1^{-1} \in M^\infty$.

**Appendix.** Let $S$ be a Blaschke sequence such that $\log p(e^{i\theta})$ is integrable. There is a function $\delta(e^{i\theta})$, of class $C^2$ on $|z| = 1$, such that $C_{1} \rho^2(e^{i\theta}) < \delta(e^{i\theta}) < \rho^2(e^{i\theta})$. In fact $\delta$ is nothing but the square of the “regularized distance” $\Delta$ to the set $S$ [6, p. 171]. Let now $D^+$ be the region bounded by the curve $r = 1 + \delta(e^{i\theta})$, so $D^+ \supseteq D$ and $D^+$ is a region of class $C^2$. There exists a conformal mapping $\Phi$ of $D^+$ onto $D$ such that $\Phi$ and $\Phi^{-1}$ both have derivatives continuous up to the boundary, and even Hölder-continuous [2, p.374]; whence $a|z_1 - z_2| < |\Phi(z_1) - \Phi(z_2)| < b|z_1 - z_2|$ for certain constants $a > 0, b > 0$ (Kellogg's theorem). The distance of $z_k$ from $\partial D^+$ is at most $1 - |z_k| + \delta(e^{i\theta})$ if $z_k = e^{i\theta}|z_k|$, so the distance is at most $2(1 - |z_k|)$. Consequently $1 - |\Phi(z_k)| = O(1 - |z_k|)$, whence $\Phi(S)$ is a Blaschke sequence in $D$, and there is a bounded analytic function $B$, on $D^+$, with zero-set $S$.

Let $\mu_\theta$ be the harmonic measure on $\partial D^+$ for the point $z$. By the differentiability properties of $\Phi$ and $\Phi^{-1}$, we see that $L^1(d\theta)$ and $L^1(d\mu_\theta)$ can be identified. Moreover, elementary geometry yields the inequality $\rho(Re^{i\theta}) > \rho(e^{i\theta})$ when $R > 1$, so that $\log \rho(w)$ belongs to $L^1(d\mu_\theta)$. By a classical method we can find a monotone function $\psi(t)$ on $t > 0$, such that $\psi(t) > t + 1$ and $\psi(t)/t \to +\infty$ as $t \to +\infty$ and $\psi(\log \rho(w))$ is in $L^1(d\mu_\theta)$. Let $u$ be its Poisson integral on $D^+$, $v$ the harmonic conjugate of $u$, and $g = e^{-u-iv}B$. 
Before completing the proof that $g$ belongs to $A^\infty$, we observe an inequality on $\mu_z$ for $z$ near $\partial D^+$. Let $\Gamma_z$ be the part of the boundary defined by the inequality $|z - w| < 3d(z, \partial D^+)$. Using the continuity of $\Phi'$ we find that $\mu_z(\Gamma_z) > a > 0$ for all $z$ in $D^+$. On $\Gamma_z$ we have $\rho(w) < \rho(z) + 3d(z, \partial D^+)$, so that, if $\rho(z) + d(z, \partial D^+)$ is small, $u(z)$ is a large multiple of

$$-\log[\rho(z) + d(z, \partial D^+)],$$

and $|g(z)|$ is bounded by a large power of $\rho + d$. Thus, for each $N > 0$

$$|g(z)| < C_N[\rho(z) + d(z, \partial D^+)]^N.$$

Around each $z$ in $D^+$ we draw a disk of radius $d(z, \partial D^+)/2$, and observe that the bound for $g$ is increased by at most $2^N$. Cauchy’s formulas give, for $r = 0, 1, 2, \ldots$,

$$|g^{(r)}(z)| \leq C_{N,r}[\rho(z) + d(z, \partial D^+)]^N/d^r(z, \partial D).$$

Let $\epsilon > 0$, and observe that on the set defined by the inequality $d(z, \partial D^+) > \epsilon p^2(z)$, each $g^{(r)}$ is uniformly bounded. But for small $\epsilon > 0$, this set contains the disk $|z| < 1$. Indeed $d(|z| e^{i\theta}, \partial D^+) > 1 - |z| + \epsilon\rho^2(e^{i\theta})$, so the inequality is true as soon as $\epsilon(1 - 2\epsilon) > \epsilon$.

In the proof just completed, $S$ could have contained points on $\partial D$, in which case all the derivatives $g^{(r)}$ vanish on $S \cap \partial D$. It is also worth remarking that a Blaschke sequence contained in a ball $|z - r| < 1$ is a zero-set for $A^\infty$, because $\rho(e^{i\theta}) > |e^{i\theta} - r| - 1 + r > a\theta^2$ for small $\theta$.

**References**


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