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AN EXPRESSION FOR THE FIRST CONJUGATE POINT FOR AN nTH ORDER NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. It is proved that the first conjugate point for an nth order nonlinear differential equation is the infimum of the first conjugate points of the corresponding variational equations.

We are concerned with the nth order nonlinear differential equation

$$y^{(n)} = f(x,y,y',\dots,y^{(n-1)}) \tag{1}$$

where we assume throughout that:

(A) f and f_i, $0 \leqslant i \leqslant n-1$, where

$$f_i(x,y,\dots,y^{(n-1)}) \equiv \frac{\partial f}{\partial y^{(i)}}(x,y,\dots,y^{(n-1)})$$

are continuous on $[a,b) \times R^n$, $a < b \leqslant \infty$.

(B) Solutions of (1) extend to $[a,b)$, and the compactness assumption.

(C) If $\{y_j(x)\}$ is a sequence of solutions of (1) which is uniformly bounded on a nondegenerate compact interval $[c,d] \subset [a,b)$, then there is a subsequence $\{y_{j_k}(x)\}$ such that $\{y_{j_k}^{(i)}(x)\}$ converges uniformly on each compact subinterval of $[a,b)$ for $i = 0,\dots,n-1$.

Let $y_0(x)$ be a solution of (1), then the linear differential equation

$$z^{(n)} = \sum_{i=0}^{n-1} f_i(x,y_0(x),\dots,y_0^{(n-1)}(x))z^{(i)} \tag{2}$$

is called [1] the variational equation of (1) along the solution $y_0(x)$.

Let $t \in [a,b)$. If there do not exist distinct solutions of (1) whose difference has n zeros, counting multiplicities, on $[t,b)$ then we say (1) is disconjugate on $[t,b)$ and we signify this by writing $\eta_1(t) = b$. If (1) is not disconjugate on $[t,b)$, then the first conjugate point, $\eta_1(t)$, of $x = t$ for (1) is the infimum of the set of $t_1 \in (t,b)$ such that there are distinct solutions of (1) whose difference has n zeros, counting multiplicities, on $[t,t_1]$. For some results concerning $\eta_1(t)$ see [8] and [9]. We will denote the first conjugate point of $x = t$ for (2) by

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$\eta_1(t; y_0(x))$. Our main result is that

$$(3) \qquad \eta_1(t) = \inf\{\eta_1(t; y_0(x)) \colon y_0(x) \text{ is a solution of (1)}\}.$$

Henceforth the right-hand side of (3) will be denoted by $\inf_{(y_0(x))} \eta_1(t; y_0(x))$. Spencer's main results in [9] were that if (A) and (B) hold, then $\eta_1(t) \leqslant \inf_{(y_0(x))} \eta_1(t; y_0(x))$ and noted that (3) holds when $n = 2, 3$. For $n = 2$ (A) and (B) imply (C), and for $n = 3$ if $\eta_1(t) > t$ on a dense subset of $[a, b)$, then (A) and (B) imply (C) (see [5]). To see that for $n = 3$ (A) and (B) do not imply (C), see [4].

Let $i_1, \ldots, i_k$ be positive integers with $\sum_{j=1}^k i_j = n$; then the boundary value problem (BVP)(1),

$$(4) \qquad y^{(m_j)}(x_j) = A_{jm_j}, \qquad j = 1, \ldots, k, \; 0 \leqslant m_j \leqslant i_j - 1,$$

is called an $(i_1, \ldots, i_k)$-BVP. Let I be a subinterval of $[a, b)$, then we say that (1) is $(i_1, \ldots, i_k)$-disconjugate on I provided for any k points $x_1 < \cdots < x_k$ in I all $(i_1, \ldots, i_k)$-BVP's (1), (4) have at most one solution. Note that (1) is disconjugate on I iff (1) is $(i_1, \ldots, i_k)$-disconjugate on I for all $k = 2, \ldots, n$, $\sum_{j=1}^k i_j = n$.

THEOREM 1. *If* (A), (B) *and* (C) *hold, then*

$$\eta_1(t) = \inf_{(y_0(x))} \eta_1(t; y_0(x)).$$

PROOF. Spencer [9] proved that $\eta_1(t) \leqslant \inf_{(y_0(x))} \eta_1(t; y_0(x))$. To prove the theorem we will show that the assumption $\eta_1(t) < \inf_{(y_0)} \eta_1(t; y_0(x))$ leads to a contradiction. First we define an ordering (lexicographic ordering) for all k-tuples $(i_1, \ldots, i_k)$ with $\sum_{j=1}^k i_j = n$, $2 \leqslant k \leqslant n$. We write $(j_1, \ldots, j_l) > (i_1, \ldots, i_k)$ if $j_1 > i_1$ or if there is an $s \in \{1, \ldots, l-1\}$ such that $j_l = i_l$, $l = 1, \ldots, s$, but $j_{s+1} > i_{s+1}$.

Since we are assuming that (1) is not disconjugate on $[t, \tau)$,

$$\tau \equiv \inf_{(y_0)} \eta_1(t; y_0(x)),$$

there is at least one tuple $(j_1, \ldots, j_l)$ such that (1) is not $(j_1, \ldots, j_l)$-disconjugate on $[t, \tau)$. Let $(i_1, \ldots, i_k)$ be the last of these tuples. By [9, Theorem 1.9], (1) is $(n-1, 1)$-disconjugate on $[t, \tau)$ and so $(i_1, \ldots, i_k) < (n-1, 1)$. Hence $i_1 < n - 1$. Since $(i_1 + 1, i_2, \ldots, i_k - 1) > (i_1, \ldots, i_k)$ (if $i_k = 1$, then by the k-tuple $(i_1 + 1, i_2, \ldots, i_k - 1)$ we mean the $(k-1)$-tuple $(i_1 + 1, i_2, \ldots, i_{k-1})$), (1) is $(i_1 + 1, i_2, \ldots, i_k - 1)$-disconjugate on $[t, \tau)$.

Since (1) is not $(i_1, \ldots, i_k)$-disconjugate on $[t, \tau)$, there are distinct solutions $y_1(x), y_2(x)$ of (1) and k points $t \leqslant x_1 < x_2 < \cdots < x_k < \tau$ such that

$$y_1^{(m_j)}(x_j) = y_2^{(m_j)}(x_j)$$

for $j = 1, \ldots, k$, $m_j = 0, \ldots, i_j - 1$. Set $s_1 = y_1^{(i_1)}(x_1)$ and $s_2 = y_2^{(i_1)}(x_1)$. By the $(i_1 + 1, i_2, \ldots, i_k - 1)$-disconjugacy, $s_1 \neq s_2$.

For s a real number consider the $(i_1 + 1, i_2, \ldots, i_k - 1)$-BVP (1)

$$y^{(m_j)}(x_j) = y_1^{(m_j)}(x_j), \qquad j = 1, \ldots, k-1,\ m_j = 0, \ldots, i_j - 1,$$

$$y^{(m_1)}(x_1) = s,$$

$$y^{(m_k)}(x_k) = y_1^{(m_k)}(x_k), \qquad m_k = 0, \ldots, i_k - 2$$

(if $i_k = 1$ there is no boundary condition at x_k). By [10, Theorem 2.9], $S = \{s \in R$: the above $(i_1 + 1, i_2, \ldots, i_k - 1)$-BVP has a solution$\}$ is an open interval (this is the place where the compactness assumption (C) is used in this proof). For each $s \in S$ let $y(x, s)$ be the solution of this BVP.

Now

$$0 = y_1^{(i_k-1)}(x_k) - y_2^{(i_k-1)}(x_k) = y^{(i_k-1)}(x_k, s_1) - y^{(i_k-1)}(x_k, s_2).$$

Since S is connected we have from [6, Theorem 8] that

$$0 = (s_1 - s_2)\frac{\partial y^{(i_k-1)}}{\partial s}(x_k, \bar{s}) = (s_1 - s_2) z^{(i_k-1)}(x_k, \bar{s})$$

where $\bar{s}$ is between s_1 and s_2 and $z(x, \bar{s}) = \partial y(x_k, \bar{s})/\partial s$ is the solution of the $(i_1 + 1, i_2, \ldots, i_k - 1)$-BVP

$$z^{(n)} = \sum_{i=0}^{n-1} f_i(x, y(x, \bar{s}), \ldots, y^{(n-1)}(x, \bar{s})) z^{(i)},$$

(5)
$$z^{(m_j)}(x_j) = 0, \qquad j = 1, \ldots, k-1,\ m_j = 0, \ldots, i_j - 1,$$

$$z^{(i_1)}(x_1) = 1,$$

$$z^{(m_k)}(x_k) = 0, \qquad m_k = 0, \ldots, i_k - 2$$

(if $i_k - 1$ there is no boundary condition at x_k).

Since $s_1 \neq s_2$ we have that $z^{(i_k-1)}(x_k, \bar{s}) = 0$. This contradicts the $(i_1, i_2, \ldots, i_k)$-disconjugacy of (5) on $[t, \tau)$ and the proof is complete.

Hence Theorem 1 reduces disconjugacy of (1) to disconjugacy of corresponding linear differential equations. There are all kinds of sufficient conditions for disconjugacy of linear differential equations. Therefore using Theorem 1, all kinds of sufficient conditions for equation (1) to be disconjugate could be given. Instead of giving any of these nice results we will illustrate Theorem 1 by a new result for $n = 4$. To this end consider the linear differential equations

(6)
$$y^{(4)} = \sum_{i=0}^{3} p_i(x) y^{(i)},$$

(7) $$y^{(4)} = \sum_{i=0}^{3} q_i(x) y^{(i)},$$

where the coefficients are assumed to be continuous.

Let $u(x,\tau;i)$, $i = 6, 7$, be the solution of the initial value problem (i)

$$y^{(j)}(\tau) = 0, \quad j = 0, 1, 2, \quad y'''(\tau) = 1.$$

For results concerning when $u^{(j)}(x,\tau;i) > 0, j = 0, 1, 2, 3, x > \tau$ and $(-1)^j u^{(j)}(x,\tau;i) < 0, j = 0, 1, 2, 3, x < \tau$, see [3] and [7].

THEOREM 2. *Assume $n = 4$ and* (A), (B) *and* (C) *hold. Assume $[c,d] \subset [a,b)$ and*

$$p_i(x) \leqslant f_i(x,y,y',y'',y'''),$$

$$(-1)^i q_i(x) \leqslant (-1)^i f_i(x,y,y',y'',y'''),$$

on $[c,d] \times R^4$ for $i = 0, 1, 2, 3$ and $f_0(x,y,y',y'',y''') \leqslant 0$ on $[c,d] \times R^4$. If $u^{(j)}(x,d;6) > 0$ on $(c,d]$ and $(-1)^j u^{(j)}(x,d;7) < 0$ on $[c,d)$ for $j = 0, 1, 2, 3$, then (1) *is disconjugate on $[c,d]$.*

PROOF. By Theorem 1 it suffices to show that all variational equations

$$z^{(4)} = \sum_{i=0}^{3} f_i(x,y_0(x),y_0'(x),y_0''(x),y'''_0(x)) z^{(i)},$$

where $y_0(x)$ is a solution of (1) with $n = 4$, are disconjugate on $[c,d]$.

Since $u(x,c;6)$ is a solution of (6) satisfying

$$u^{(j)}(x,c;6) > 0 \quad \text{on } (c,d], \qquad j = 0, 1, 2, 3,$$

it follows easily that there is a solution $y_3(x)$ of (6) satisfying

$$y_3^{(j)}(x) > 0 \quad \text{on } [c,d], \qquad j = 0, 1, 2, 3.$$

Similarly, since $-u(x,d;7)$ is a solution of (7) satisfying

$$(-1)^j y^{(j)}(x) > 0 \quad \text{on } [c,d), \qquad j = 0, 1, 2, 3,$$

there is a solution $y_1(x)$ of (7) satisfying

$$(-1)^j y_1^{(j)}(x) > 0 \quad \text{on } [c,d], \qquad j = 0, 1, 2, 3.$$

Set $y_2(x) \equiv 1$ on $[c,d]$. Then it is easy to show that the Wronskians $W[y_1,y_2]$, $W[y_1,y_3]$, $W[y_2,y_3]$, and $W[y_1,y_2,y_3]$ are positive on $[c,d]$. Define L on $C^4[c,d]$ by

$$Lz(x) = z^{(4)}(x) - \sum_{j=0}^{3} f_j(x,y_0(x),y_0'(x),y_0''(x),y'''_0(x)) z^{(j)}(x).$$

It is easy to check that

$$Ly_1(x) \leqslant 0,$$
$$Ly_2(x) = -f_0(x, y_0(x), y_0'(x), y_0''(x), y'''_0(x)) \geqslant 0,$$
$$Ly_3(x) \leqslant 0$$

on $[c, d]$. Hence by [2, Proposition 4.1] $Lz = 0$ is disconjugate on $[c, d]$ and the proof is complete.

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